Homework11 答案(2020)

1.证明Heisenberg Picture下自旋角动量对易式$[S_x^{(H)}, S_y^{(H)}] = i\hbar S_z^{(H)}$.如果$S_y^{(H)}(t) = S_y^{(S)} cos(\omega t) - S_z^{(S)} sin(\omega t)$，初态$|\alpha(0)>=|+,z>$，请问测量$S_y(t)$的可能测值和相应概率的时间演化。$S_y^{(H)}$的本征态是什么?

解:

$$\begin{aligned}
[S_x^{(H)}, S_y^{(H)}] &= [e^{iHt}S_x e^{-iHt}, e^{iHt}S_y e^{-iHt}] \\
&= e^{iHt}[S_x, S_y]e^{-iHt} \\
&= e^{iHt}(i\hbar S_z)e^{-iHt} \\
&= i\hbar S_z^{(H)}
\end{aligned} \tag{1}$$

上式得证.

设S_y的本征值为$\frac{\hbar}{2}\lambda$,由泡利矩阵可以写出:

$$\frac{\hbar}{2}\begin{pmatrix} -\sin\omega t & -i\cos\omega t \\ i\cos\omega t & \sin\omega t \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \frac{\hbar}{2}\lambda\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \tag{2}$$

则久期方程为:

$$det\begin{vmatrix} -sin\omega t - \lambda & -icos\omega t \\ icos\omega t & sin\omega t - \lambda \end{vmatrix} = 0 \tag{3}$$

解得$\lambda = \pm 1$.

对于$\lambda = 1$,即本征值为$\frac{\hbar}{2}$时，解得本征态为:

$$\psi_+ = \frac{1}{\sqrt{2+2sin\omega t}}\begin{pmatrix} cos\omega t \\ i(1+sin\omega t) \end{pmatrix} \tag{4}$$

对于$\lambda = -1$,即本征值为$-\frac{\hbar}{2}$时，解得本征态为:

$$\psi_- = \frac{1}{\sqrt{2-2sin\omega t}}\begin{pmatrix} cos\omega t \\ i(sin\omega t - 1) \end{pmatrix} \tag{5}$$

对应的概率为:

$$p_+ = |<\psi_+|\alpha(0)>|^2 \tag{6}$$
$$p_- = |<\psi_-|\alpha(0)>|^2 \tag{7}$$

将求得的本征态带入上面两个式子，得到:

$$p_+ = \frac{cos^2\omega t}{2+2sin\omega t} \tag{8}$$

$$p_- = \frac{cos^2\omega t}{2-2sin\omega t} \tag{9}$$

2. 质量为m电荷q的粒子在匀强电场中运动（场强ϵ）。已知$t = 0$时$\langle x\rangle = 0, \langle p_x\rangle = p_0$。利用Heisenberg方程计算$\langle x(t)\rangle, \langle p_x(t)\rangle$

解: 哈密顿量为$H = \frac{p_x^2}{2m} - \epsilon q x$, 得到$x, p$的Heisenberg运动方程为

$$\frac{dx(t)}{dt} = \frac{1}{i\hbar}[x(t), H] = \frac{p_x(t)}{m} \tag{10}$$

$$\frac{dp_x(t)}{dt} = \frac{1}{i\hbar}[p_x(t), H] = \epsilon q \tag{11}$$

解得

$$\begin{aligned} p_x(t) &= p_x(0) + \epsilon q t \\ x(t) &= x(0) + \frac{1}{m}(p_x(0)t + \frac{1}{2}\epsilon q t^2) \end{aligned} \tag{12}$$

$x(t), p_x(t)$的期望值为

$$\begin{aligned} \langle p_x(t)\rangle &= \langle p_x(0)\rangle + \epsilon q t \\ &= p_0 + \epsilon q t \end{aligned} \tag{13}$$

$$\begin{aligned} \langle x(t)\rangle &= \langle x(0)\rangle + \frac{1}{m}(\langle p_x(0)\rangle t + \frac{1}{2}\epsilon q t^2) \\ &= p_0 t + \frac{1}{2m}\epsilon q t^2 \end{aligned} \tag{14}$$

3. 带电粒子在z方向磁场中运动。哈密顿量近似为$H = p^2/2m - \omega L_z, \omega = qB/(2mc)$. (a)已知$t = 0$时刻，$\langle\vec{p}\rangle = (p_0, 0, 0)$, 求$t > 0$时$\langle\vec{p}(t)\rangle$. (b) 指出守恒量。

解：此题描述的是磁场中带电粒子的运动，类似于经典粒子受Lorentz力作用。

(a) 由Heisenberg 方程得

$$\begin{aligned} \frac{dp_x(t)}{dt} &= \frac{1}{i\hbar}[p_x(t), H] \\ &= -\frac{\omega}{i\hbar}[p_x(t), L_z] \\ &= \omega p_y(t) \end{aligned} \tag{15}$$

$$\frac{dp_y(t)}{dt} = \frac{1}{i\hbar}[p_y(t), H] = -\omega p_x(t) \tag{16}$$

$$\frac{dp_z(t)}{dt} = \frac{1}{i\hbar}[p_z(t), H] = 0 \tag{17}$$

由以上方程得到微分方程

$$\frac{d^2p_x(t)}{dt^2} = -\omega^2 p_x(t) \tag{18}$$

$$\frac{d^2p_y(t)}{dt^2} = -\omega^2 p_y(t) \tag{19}$$

$$\frac{dp_z(t)}{dt} = 0 \tag{20}$$

和初始条件

$$\frac{dp_x(t)}{dt}|_{t=0} = \omega p_y(0) \tag{21}$$

$$\frac{dp_y(t)}{dt}|_{t=0} = -\omega p_x(0) \tag{22}$$

再利用$\langle\vec{p}\rangle = (p_0, 0, 0)$，解得

$$p_x(t) = p_0 cos\omega t \tag{23}$$

$$p_y(t) = -p_0 sin\omega t \tag{24}$$

$$p_z(t) = 0 \tag{25}$$

$\vec{p}(t)$的期望值为

$$\langle\vec{p}(t)\rangle = (cos\omega t p_0, -sin\omega t p_0, 0) \tag{26}$$

(b) 由于

$$[p_z, H] = [L_z, H] = [T, H] = [\vec{L}^2, H] = 0 \tag{27}$$

故守恒量有$p_z, L_z, T, \vec{L}^2$

4. 对于一维谐振子，(a) 处于能量本征态$|n\rangle$。利用代数方法求动能期望值和势能期望值，及其标准差。(b) 初态$\psi(0) = \frac{1}{\sqrt{2}}\psi_0 + \frac{1}{\sqrt{2}}\psi_1$, 在S-pic和H-pic下分别计算$\langle p(t)\rangle$。
解：(a)利用$a, a^\dagger$表示$\hat{x}, \hat{p}$

$$\hat{x} = (\frac{\hbar}{2m\omega})^{\frac{1}{2}}(a + a^\dagger) \tag{28}$$

$$\hat{p} = i(\frac{m\hbar\omega}{2})^{\frac{1}{2}}(a^\dagger - a) \tag{29}$$

把式(1),(2)代入动能项$\hat{T}$和势能项$\hat{V}$得到

$$\begin{aligned}\hat{T} &= \frac{\hat{p}^2}{2m} \\ &= -\frac{\hbar\omega}{4}(a - a^\dagger)(a - a^\dagger) \\ &= -\frac{\hbar\omega}{4}(a^2 + (a^\dagger)^2 - a^\dagger a - aa^\dagger) \\ &= -\frac{\hbar\omega}{4}(a^2 + (a^\dagger)^2 - 2a^\dagger a - 1) \\ &= -\frac{\hbar\omega}{4}(a^2 + (a^\dagger)^2 - 2\hat{n} - 1)\end{aligned} \tag{30}$$

$$
\begin{aligned}
\hat{V} &= \frac{1}{2}m\omega^2\hat{x}^2 \\
&= \frac{\hbar\omega}{4}(a+a^\dagger)(a+a^\dagger) \\
&= \frac{\hbar\omega}{4}(a^2+(a^\dagger)^2+a^\dagger a+aa^\dagger) \\
&= \frac{\hbar\omega}{4}(a^2+(a^\dagger)^2+2a^\dagger a+1) \\
&= \frac{\hbar\omega}{4}(a^2+(a^\dagger)^2+2\hat{n}+1) \qquad (31)
\end{aligned}
$$

又$< n|a^2|n >=< n|(a^\dagger)^2|n >=0$.因此，

$$
\begin{aligned}
\langle n|\hat{T}|n\rangle &= -\frac{\hbar\omega}{4}\langle n|a^2+(a^\dagger)^2-2\hat{n}-1|n\rangle \\
&= -\frac{\hbar\omega}{4}\langle n|-2\hat{n}-1|n\rangle \\
&= \frac{\hbar\omega(2n+1)}{4} \qquad (32) \\
\langle n|\hat{V}|n\rangle &= \frac{\hbar\omega}{4}\langle n|a^2+(a^\dagger)^2+2\hat{n}+1|n\rangle \\
&= \frac{\hbar\omega}{4}\langle n|2\hat{n}+1|n\rangle \\
&= \frac{\hbar\omega(2n+1)}{4} \qquad (33) \\
\langle n|\hat{T}^2|n\rangle &= (\frac{\hbar\omega}{4})^2\langle n|(a^2+(a^\dagger)^2-2\hat{n}-1)^2|n\rangle \\
&= (\frac{\hbar\omega}{4})^2\langle n|a^2(a^\dagger)^2+(a^\dagger)^2a^2+(2\hat{n}+1)^2|n\rangle \\
&= (\frac{(2n+1)\hbar\omega}{4})^2+(\frac{\hbar\omega}{4})^2\langle n|a(aa^\dagger)a^\dagger+a^\dagger(a^\dagger a)a|n\rangle \\
&= (\frac{(2n+1)\hbar\omega}{4})^2+(\frac{\hbar\omega}{4})^2\langle n|a(a^\dagger a+1)a^\dagger+a^\dagger\hat{n}a|n\rangle \\
&= (\frac{(2n+1)\hbar\omega}{4})^2+(\frac{\hbar\omega}{4})^2\langle n|a\hat{n}a^\dagger+aa^\dagger+a^\dagger\hat{n}a|n\rangle \\
&= (\frac{(2n+1)\hbar\omega}{4})^2+(\frac{\hbar\omega}{4})^2\langle n|(n+1)^2+n+1+(n-1)n|n\rangle \\
&= (\frac{(2n+1)\hbar\omega}{4})^2+(\frac{\hbar\omega}{4})^2\langle n|2n^2+2n+2|n\rangle \\
&= (\frac{\hbar\omega}{4})^2(6n^2+6n+3) \qquad (34)
\end{aligned}
$$

同理，

$$
\begin{aligned}
\langle n|\hat{V}^2|n\rangle &= (\frac{\hbar\omega}{4})^2\langle n|(a^2+(a^\dagger)^2+2\hat{n}+1)^2|n\rangle \\
&= (\frac{\hbar\omega}{4})^2(2n+1)^2+(\frac{\hbar\omega}{4})^2\langle n|a^2(a^\dagger)^2+(a^\dagger)^2a^2|n\rangle
\end{aligned}
$$

$$= \quad (\frac{\hbar\omega}{4})^2(6n^2+6n+3) \tag{35}$$

由以上结果得到其标准偏差

$$\begin{aligned} \Delta T &= \sqrt{<T^2> - <T>^2} \\ &= \frac{\hbar\omega}{4}\sqrt{2n^2+2n+2} \end{aligned} \tag{36}$$

$$\begin{aligned} \Delta V &= \sqrt{<V^2> - <V>^2} \\ &= \frac{\hbar\omega}{4}\sqrt{2n^2+2n+2} \end{aligned} \tag{37}$$

(b) 在Schrodinger picture下，我们很容易可以得到t时刻的波函数

$$\begin{aligned} |\psi(t)\rangle &= \frac{1}{\sqrt{2}}e^{-\frac{i}{\hbar}E_0 t}|0\rangle + \frac{1}{\sqrt{2}}e^{-\frac{i}{\hbar}E_1 t}|1\rangle \\ &= \frac{1}{\sqrt{2}}e^{-i\frac{1}{2}\omega t}|0\rangle + \frac{1}{\sqrt{2}}e^{-i\frac{3}{2}\omega t}|1\rangle \end{aligned} \tag{38}$$

由t时刻的波函数我们可以计算

$$\begin{aligned} <p> &= \langle\psi(t)|p|\psi(t)\rangle \\ &= \frac{1}{2}\langle 0|p|0\rangle + \frac{1}{2}\langle 1|p|1\rangle + \frac{1}{2}e^{i\omega t}\langle 1|p|0\rangle + \frac{1}{2}e^{-i\omega t}\langle 0|p|1\rangle \\ &= \frac{1}{2}e^{i\omega t}\langle 1|p|0\rangle + \frac{1}{2}e^{-i\omega t}\langle 0|p|1\rangle \\ &= i(\frac{m\hbar\omega}{2})^{\frac{1}{2}}\frac{1}{2}(e^{i\omega t}\langle 1|(a^\dagger - a)|0\rangle + e^{-i\omega t}\langle 0|(a^\dagger - a)|1\rangle) \\ &= i(\frac{m\hbar\omega}{2})^{\frac{1}{2}}\frac{1}{2}(e^{i\omega t} - e^{-i\omega t}) \\ &= -(\frac{m\hbar\omega}{2})^{\frac{1}{2}} sin\omega t \end{aligned} \tag{39}$$

在Heisenberg picture下，由Heisenberg 运动方程得

$$\begin{aligned} \frac{d\hat{p}(t)}{dt} &= \frac{1}{i\hbar}[\hat{p}(t), \hat{H}(t)] \\ &= -m\omega^2\hat{x}(t) \end{aligned} \tag{40}$$

$$\begin{aligned} \frac{d\hat{x}(t)}{dt} &= \frac{1}{i\hbar}[\hat{x}(t), \hat{H}(t)] \\ &= \frac{1}{m}\hat{p}(t) \end{aligned} \tag{41}$$

我们可以得到关于$p(t)$的二阶微分方程

$$\frac{d^2\hat{p}(t)}{dt^2} = -\omega^2\hat{p}(t) \tag{42}$$

$$\frac{dp(t)}{dt}|_{t=0} = -m\omega^2\hat{x}(0) \tag{43}$$

解得

$$\hat{p}(t) = -m\omega^2 \hat{x}(0) sin\omega t + \hat{p}(0) cos\omega t \tag{44}$$

$\hat{p}(t)$期望值为

$$\begin{aligned}
\langle \hat{p}(t) \rangle &= \langle \psi(0)|\hat{p}(t)|\psi(0)\rangle \\
&= -m\omega^2 sin\omega t \langle \psi(0)|x(0)|\psi(0)\rangle + cos\omega t \langle \psi(0)|p(0)|\psi(0)\rangle \\
&= -m\omega^2 sin\omega t (\frac{\hbar}{2m\omega})^{\frac{1}{2}} + 0 \\
&= -(\frac{m\hbar\omega}{2})^{\frac{1}{2}} sin\omega t
\end{aligned} \tag{45}$$

5. 设谐振子初态为相干态$|\alpha\rangle$. 请问(a) 计算量子数算符$a^\dagger a$的期望值和标准偏差。（b）计算动量的期望值随时间变化情况和标准偏差。在什么情况下此标准偏差可以忽略?
解:

$$\begin{aligned}
\langle \alpha|a^\dagger a|\alpha\rangle &= \langle \alpha|\alpha^* \alpha|\alpha\rangle \\
&= |\alpha|^2
\end{aligned} \tag{46}$$

$$\begin{aligned}
\langle \alpha|(a^\dagger a)^2|\alpha\rangle &= \langle \alpha|a^\dagger a a^\dagger a|\alpha\rangle \\
&= \langle \alpha|\alpha^*(a^\dagger a + 1)\alpha|\alpha\rangle \\
&= |\alpha|^2 + \langle \alpha|\alpha^* \alpha^* \alpha\alpha|\alpha\rangle \\
&= |\alpha|^4 + |\alpha|^2
\end{aligned} \tag{47}$$

$$\begin{aligned}
\Delta(a^\dagger a) &= \sqrt{\langle \alpha|(a^\dagger a)^2|\alpha\rangle - \langle \alpha|a^\dagger a|\alpha\rangle^2} \\
&= |\alpha|
\end{aligned} \tag{48}$$

(b) t时刻相干态为

$$\begin{aligned}
|\alpha(t)\rangle &= e^{-\frac{|\alpha|^2}{2}} e^{-i\frac{\omega}{2}t} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} e^{-i\omega n t}|n\rangle \\
a|\alpha(t)\rangle &= e^{-i\omega t}\alpha|\alpha(t)\rangle \\
\langle \alpha(t)|a^\dagger &= \langle \alpha(t)|e^{i\omega t}\alpha^*
\end{aligned} \tag{49}$$

得到

$$\begin{aligned}
\langle \alpha(t)|p|\alpha(t)\rangle &= -i(\frac{m\hbar\omega}{2})^{\frac{1}{2}} \langle \alpha(t)|(a - a^\dagger)|\alpha(t)\rangle \\
&= -i(\frac{m\hbar\omega}{2})^{\frac{1}{2}} (e^{-i\omega t}\alpha - e^{i\omega t}\alpha^*) \\
&= -i(\frac{m\hbar\omega}{2})^{\frac{1}{2}} |\alpha|(e^{i(\theta-\omega t)} - e^{i(\omega t-\theta)}),\ with\ \alpha = |\alpha|e^{i\theta} \\
&= 2(\frac{m\hbar\omega}{2})^{\frac{1}{2}} |\alpha| sin(\theta - \omega t)
\end{aligned} \tag{50}$$

$$\begin{aligned}\langle\alpha(t)|p^2|\alpha(t)\rangle &= -(\frac{m\hbar\omega}{2})\langle\alpha(t)|(a-a^\dagger)^2|\alpha(t)\rangle \\ &= -(\frac{m\hbar\omega}{2})\langle\alpha(t)|a^2+(a^\dagger)^2-2a^\dagger a-1|\alpha(t)\rangle \\ &= -(\frac{m\hbar\omega}{2})(e^{-i2\omega t}\alpha^2+e^{i2\omega t}(\alpha^*)^2-2|\alpha|^2-1) \\ &= (\frac{m\hbar\omega}{2})(1+2|\alpha|^2-2cos(2\theta-2\omega t)|\alpha|^2) \\ &= \frac{m\hbar\omega}{2}+[2(\frac{m\hbar\omega}{2})^{\frac{1}{2}}|\alpha|sin(\theta-\omega t)]^2 \qquad (51)\end{aligned}$$

则

$$\Delta p = (\frac{m\hbar\omega}{2})^{\frac{1}{2}} \qquad (52)$$

那么

$$\frac{\Delta p}{p} = \frac{1}{2|\alpha|sin(\theta-\omega t)} \qquad (53)$$

当$|\alpha|\gg 1$时，相对误差很小，此标准偏差可以忽略。

6. 考虑轨道角动量。限定$l=1$, 取基矢为$|1,1\rangle,|1,0\rangle,|1,-1\rangle$. （a）利用代数方法求$L_x,L_y$矩阵表示及本征值和本征矢量。（b）将$L^2,L_x$的共同本征函数表示成$Y_{lm}$的线性叠加。
解：令

$$|1,1\rangle=\begin{pmatrix}1\\0\\0\end{pmatrix},|1,0\rangle=\begin{pmatrix}0\\1\\0\end{pmatrix},|1,-1\rangle=\begin{pmatrix}0\\0\\1\end{pmatrix} \qquad (54)$$

利用

$$L_\pm|l,m\rangle = \hbar\sqrt{l(l+1)-m(m\pm 1)}|l,m\pm 1\rangle \qquad (55)$$

$$L_\pm = L_x \pm iL_y \qquad (56)$$

则

$$L_x=\frac{1}{2}(L_++L_-) \qquad L_y=\frac{1}{2i}(L_++L_-) \qquad (57)$$

可以分别计算L_x,L_y的矩阵元

$$\begin{matrix}(L_x)_{11}=\langle 1,1|L_x|1,1\rangle, & (L_x)_{12}=\langle 1,1|L_x|1,0\rangle, & (L_x)_{13}=\langle 1,1|L_x|1,-1\rangle \\ (L_x)_{21}=(L_x)^*_{12}, & (L_x)_{22}=\langle 1,-1|L_x|1,-1\rangle, & (L_x)_{23}=\langle 1,0|L_x|1,-1\rangle \\ (L_x)_{31}=(L_x)^*_{13}, & (L_x)_{32}=(L_x)^*_{23}, & (L_x)_{33}=\langle 1,-1|L_x|1,-1\rangle\end{matrix}$$

$$\begin{matrix}(L_y)_{11}=\langle 1,1|L_y|1,1\rangle, & (L_y)_{12}=\langle 1,1|L_y|1,0\rangle, & (L_y)_{13}=\langle 1,1|L_y|1,-1\rangle \\ (L_y)_{21}=(L_y)^*_{12}, & (L_y)_{22}=\langle 1,-1|L_y|1,-1\rangle, & (L_y)_{23}=\langle 1,0|L_y|1,-1\rangle \\ (L_y)_{31}=(L_y)^*_{13}, & (L_y)_{32}=(L_y)^*_{23}, & (L_y)_{33}=\langle 1,-1|L_y|1,-1\rangle\end{matrix}$$

容易求得对角上的矩阵元均为0，非对角元

$$\begin{aligned}
(L_x)_{12} &= \frac{1}{2}\langle 1,1|L_+ + L_-|1,0\rangle = \frac{\sqrt{2}}{2}\hbar \\
(L_x)_{13} &= \frac{1}{2}\langle 1,1|L_+ + L_-|1,-1\rangle = 0 \\
(L_x)_{21} &= \frac{\sqrt{2}}{2}\hbar \\
(L_x)_{23} &= \frac{1}{2}\langle 1,0|L_+ + L_-|1,-1\rangle = \frac{\sqrt{2}}{2}\hbar \\
(L_x)_{31} &= 0 \\
(L_x)_{32} &= \frac{\sqrt{2}}{2}\hbar \\
(L_y)_{12} &= \frac{1}{2i}\langle 1,1|L_+ - L_-|1,0\rangle = -i\frac{\sqrt{2}}{2}\hbar \\
(L_y)_{13} &= \frac{1}{2i}\langle 1,1|L_+ - L_-|1,-1\rangle = 0 \\
(L_y)_{21} &= i\frac{\sqrt{2}}{2}\hbar \\
(L_y)_{23} &= \frac{1}{2i}\langle 1,0|L_+ - L_-|1,-1\rangle = -i\frac{\sqrt{2}}{2}\hbar \\
(L_y)_{31} &= 0 \\
(L_y)_{32} &= i\frac{\sqrt{2}}{2}\hbar
\end{aligned}$$

最后求得

$$L_x = \frac{\sqrt{2}}{2}\hbar\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$L_y = \frac{\sqrt{2}}{2}\hbar\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

设L_x的本征矢为$\phi_x = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, 解$L_x$的本征方程

$$\frac{\sqrt{2}}{2}\hbar\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \lambda\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

得到

$$\lambda_{x1} = \hbar, \quad \phi_{x1} = \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$$

$$\lambda_{x2} = 0, \quad \phi_{x2} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$\lambda_{x3} = -\hbar, \quad \phi_{x3} = \frac{1}{2}\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$$

同理求得L_y的本征值和本征矢为

$$\lambda_{y1} = \hbar, \quad \phi_{y1} = \frac{1}{2}\begin{pmatrix} 1 \\ i\sqrt{2} \\ -1 \end{pmatrix}$$

$$\lambda_{y2} = 0, \quad \phi_{y2} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\lambda_{y3} = -\hbar, \quad \phi_{y3} = \frac{1}{2}\begin{pmatrix} 1 \\ -i\sqrt{2} \\ -1 \end{pmatrix}$$

(b) 由(a)结果可知

$$\begin{aligned}
\phi_{x1} &= \frac{1}{2}|1,1\rangle + \frac{\sqrt{2}}{2}|1,0\rangle + \frac{1}{2}|1,-1\rangle \\
\phi_{x2} &= \frac{1}{\sqrt{2}}|1,1\rangle - \frac{1}{\sqrt{2}}|1,-1\rangle \\
\phi_{x3} &= \frac{1}{2}|1,1\rangle - \frac{\sqrt{2}}{2}|1,0\rangle + \frac{1}{2}|1,-1\rangle
\end{aligned}$$

7.考虑谐振子升降算符a, $a^\dagger$的线性变换

$$b = \lambda a + \nu a^\dagger$$

λ, ν为实数，满足$\lambda^2 - \nu^2 = 1$.证明：对于b的任意本征态$|\beta >$有$\Delta x \cdot \Delta y = \hbar/2$.
(提示：证明$[b, b^\dagger] = 1$，再用b，$b^\dagger$表示x, p).
证明：

$$\begin{aligned}
[b, b^+] &= [\lambda a + \nu a^+, \lambda a^+ + \nu a] \\
&= (\lambda a + \nu a^+)(\lambda a^+ + \nu a) - (\lambda a^+ + \nu a)(\lambda a + \nu a^+) \\
&= (\lambda^2 - \nu^2)[a, a^+] \\
&= 1 \qquad (58)
\end{aligned}$$

逆变换得到$a = \lambda b - \nu b^+$,$a^+ = \lambda b^+ - \nu b$，代入$x = \frac{1}{\sqrt{2}\alpha}(a + a^+)$及$p = \frac{i\hbar\alpha}{\sqrt{2}}(a^+ - a)$，有

$$x = \frac{1}{\sqrt{2}\alpha}(\lambda - \nu)(b + b^+) \qquad (59)$$

$$p = \frac{i\hbar\alpha}{\sqrt{2}}(\lambda + \nu)(b^+ - b) \qquad (60)$$

对b的本征态$|\beta>$，$b|\beta>=\beta|\beta>$

$$\begin{aligned}<x> &= \frac{1}{\sqrt{2}\alpha}(\lambda-\nu)<\beta|b+b^{+}|\beta> \\ &= \frac{1}{\sqrt{2}\alpha}(\lambda-\nu)(\beta+\beta^{*}) \qquad (61)\\ <x^2> &= \frac{1}{2\alpha^2}(\lambda-\nu)^2<\beta|b^2+b^{+2}+bb^{+}+b^{+}b|\beta> \\ &= \frac{1}{2\alpha^2}(\lambda-\nu)^2<\beta|b^2+b^{+2}+2b^{+}b+1|\beta> \\ &= \frac{1}{2\alpha^2}(\lambda-\nu)^2(\beta^2+\beta^{*2}+2|\beta|^2+1) \qquad (62)\\ <p> &= \frac{i\hbar\alpha}{\sqrt{2}}(\lambda+\nu)<\beta|b^{+}-b|\beta> \\ &= \frac{i\hbar\alpha}{\sqrt{2}}(\lambda+\nu)(\beta^{*}-\beta) \qquad (63)\\ <p^2> &= -\frac{\hbar^2\alpha^2}{2}(\lambda+\nu)^2<\beta|b^2+b^{+2}-bb^{+}-b^{+}b|\beta> \\ &= -\frac{\hbar^2\alpha^2}{2}(\lambda+\nu)^2<\beta|b^2+b^{+2}-2b^{+}b-1|\beta> \\ &= -\frac{\hbar^2\alpha^2}{2}(\lambda+\nu)^2(\beta^2+\beta^{*2}-2|\beta|^2-1) \qquad (64)\end{aligned}$$

则

$$\Delta x = \sqrt{<x^2>-<x>^2} = \frac{1}{\sqrt{2}\alpha}(\lambda-\nu) \qquad (65)$$

$$\Delta p = \sqrt{<p^2>-<p>^2} = \frac{\hbar\alpha}{\sqrt{2}}(\lambda+\nu) \qquad (66)$$

所以

$$\Delta x\cdot\Delta p = \frac{\hbar}{2} \qquad (67)$$

8. 某体系能量算符为$H=\frac{5}{3}a^{\dagger}a+\frac{2}{3}(a^2+(a^{\dagger})^2), [a,a^{\dagger}]=1$. 求体系的能谱（所有能量本征态）和基态波函数。（提示：引入$b=ua+va^{\dagger}$, 使得$[b,b^{\dagger}]=1$, 将H表示成$b^{\dagger}b$的函数）
解：令$b=ua+va^{\dagger}, b^{\dagger}=ua^{\dagger}+va$, 其中$u,v$为实数。计算$b,b^{\dagger}$的对易关系

$$\begin{aligned}[b,b^{\dagger}] &= [ua+va^{\dagger},ua^{\dagger}+va] \\ &= [ua,ua^{\dagger}]+[va^{\dagger},va] \\ &= u^2-v^2 \qquad (68)\end{aligned}$$

为使$[b,b^{\dagger}]=1$, 要求

$$u^2-v^2=1 \qquad (69)$$

用$b, b^\dagger$来表示$a, a^\dagger$

$$\begin{aligned} a &= ub - vb^\dagger \quad (70) \\ a^\dagger &= ub^\dagger - vb \quad (71) \end{aligned}$$

代入哈密顿量中得到

$$\begin{aligned} H &= \frac{5}{3}(ub^\dagger - vb)(ub - vb^\dagger) + \frac{2}{3}(ub - vb^\dagger)^2 + \frac{2}{3}(ub^\dagger - vb)^2 \\ &= \frac{5}{3}(u^2b^\dagger b + v^2bb^\dagger - uv(b^\dagger)^2 - vub^2) + \frac{2}{3}(v^2(b^\dagger)^2 + u^2b^2 - uvb^\dagger b - uvbb^\dagger) + \\ &\quad \frac{2}{3}(u^2(b^\dagger)^2 + v^2b^2 - uvb^\dagger b - uvbb^\dagger) \\ &= [\frac{5}{3}(u^2+v^2) - \frac{8}{3}uv]b^\dagger b + [\frac{2}{3}(u^2+v^2) - \frac{5}{3}uv][(b^\dagger)^2 + b^2] + \frac{5}{3}v^2 - \frac{4}{3}uv \quad (72) \end{aligned}$$

为了消除第二项，要求

$$\frac{2}{3}(u^2+v^2) - \frac{5}{3}uv = 0 \quad (73)$$

结合式69,得到$u = \sqrt{\frac{4}{3}}, v = \sqrt{\frac{1}{3}}$, 哈密顿量化简为

$$H = b^\dagger b - \frac{1}{3} \quad (74)$$

对应能量本征态为$E_k = k - \frac{1}{3}, k = 0, 1, 2....$

在坐标表象下$a = \frac{1}{\sqrt{2}}(Q + iP)$, 那么$b = \frac{1}{\sqrt{2}}(\sqrt{3}Q + \frac{1}{\sqrt{3}}P)$ 将$P = -i\frac{\partial}{\partial Q}$ 带入$\langle Q|b|\psi_0\rangle = 0$, 可以直接求出基态波函数$\psi_0(Q) = Cexp^{-3Q^2/2}$

也可以抽象地将$|k=0\rangle$ 表示成$a^\dagger a$的本征态$|n\rangle$的叠加。

设基态波函数为$|\psi_0\rangle = \sum_{n=0} c_n|n\rangle, \sum_n |c_n|^2 = 1$, 则有

$$\begin{aligned} 0 &= b|\psi_0\rangle \\ &= (ua + va^\dagger)(\sum_{n=0} c_n|n\rangle) \\ &= \sum_{n=1} c_n u\sqrt{n}|n-1\rangle + \sum_{n=0} c_n v\sqrt{n+1}|n+1\rangle \\ &= c_1u|0\rangle + \sum_{n=1}(uc_{n+1}\sqrt{n+1}|n\rangle + vc_{n-1}\sqrt{n})|n\rangle \quad (75) \end{aligned}$$

将$u = \sqrt{\frac{4}{3}}, v = \sqrt{\frac{1}{3}}$代入，我们得到

$$\begin{aligned} c_n &= 0, \qquad n = 1, 3, 5... \\ 2c_{n+2}\sqrt{n+2} + c_n\sqrt{n+1} &= 0, \qquad n = 0, 2, 4... \end{aligned}$$

解得

$$c_n = \begin{cases} 0, & n = 1, 3, 5... \\ c_0\sqrt{\frac{(n-1)!!}{n!!}}\frac{1}{2^{n/2}}, & n = 2, 4, 6... \end{cases} \quad (76)$$

$$|\psi\rangle = c_0 \sum_{n=0,2,4\ldots} \sqrt{\frac{(n-1)!!}{n!!}} \frac{1}{2^{n/2}} |n\rangle \tag{77}$$