Homework1 答案

1. 证明: 如果$\Psi(x)$是归一化的, 那么动量表象波函数$\varphi(p)$也是.

证1: 由于$\Psi(\vec{x}) = \frac{1}{(2\pi\hbar)^{3/2}}\int d^3\vec{p}\varphi(\vec{p})e^{\frac{i}{\hbar}\vec{p}\cdot\vec{x}}$

$$\begin{aligned}
1 &= \int d^3\vec{x}|\Psi(\vec{x})|^2 \\
&= \int d^3\vec{x}\Psi(\vec{x})\Psi^*(\vec{x}) \\
&= \int d^3\vec{x}\frac{1}{(2\pi\hbar)^{3/2}}\int d^3\vec{p}\varphi(\vec{p})e^{\frac{i}{\hbar}\vec{p}\cdot\vec{x}}\cdot\frac{1}{(2\pi\hbar)^{3/2}}\int d^3\vec{p\prime}\varphi^*(\vec{p\prime})e^{\frac{-i}{\hbar}\vec{p\prime}\cdot\vec{x}} \\
&= \int d^3\vec{p}\int d^3\vec{p\prime}\varphi(\vec{p})\varphi^*(\vec{p\prime})\frac{1}{(2\pi\hbar)^3}\int d^3\vec{x}e^{\frac{i}{\hbar}(\vec{p}-\vec{p\prime})\cdot\vec{x}} \\
&= \int d^3\vec{p}\int d^3\vec{p\prime}\varphi(\vec{p})\varphi^*(\vec{p\prime})\delta(\vec{p}-\vec{p\prime}) \\
&= \int d^3\vec{p}\varphi(\vec{p})\varphi^*(\vec{p}) \\
&= \int d^3\vec{p}|\varphi(\vec{p})|^2
\end{aligned}$$

证2: 利用$\varphi(\vec{p}) = \frac{1}{(2\pi\hbar)^{3/2}}\int d^3\vec{x}\Psi(\vec{x})e^{\frac{-i}{\hbar}\vec{p}\cdot\vec{x}}$

$$\begin{aligned}
\int d^3\vec{p}|\varphi(\vec{p})|^2 &= \int d^3\vec{p}\frac{1}{(2\pi\hbar)^{3/2}}\int d^3\vec{x}\Psi(\vec{x})e^{-\frac{i}{\hbar}\vec{p}\cdot\vec{x}}\cdot\frac{1}{(2\pi\hbar)^{3/2}}\int d^3\vec{x\prime}\Psi^*(\vec{x\prime})e^{\frac{i}{\hbar}\vec{p}\cdot\vec{x\prime}} \\
&= \int d^3\vec{x}\int d^3\vec{x\prime}\Psi(\vec{x})\Psi^*(\vec{x\prime})\frac{1}{(2\pi\hbar)^3}\int d^3\vec{p}e^{\frac{i}{\hbar}\vec{p}\cdot(\vec{x\prime}-\vec{x})} \\
&= \int d^3\vec{x}\int d^3\vec{x\prime}\Psi(\vec{x})\Psi^*(\vec{x\prime})\delta(\vec{x\prime}-\vec{x}) \\
&= \int d^3\vec{x}\Psi(\vec{x})\Psi^*(\vec{x}) \\
&= 1
\end{aligned}$$

2. 已知粒子的波函数$\Psi(x) = Aexp[-\frac{\alpha^2x^2}{2}+i\frac{p_0x}{\hbar}]$, (1) 计算位置统计平均(期望值); (2) 求其动量表象波函数$\varphi(p)$. (3) 计算其动量统计平均(又称期望值)和不确定度(标准差); (4) 计算其动能期望值.

解:(1)

$$\begin{aligned} <x> &= \int_{-\infty}^{\infty} dx\ x\Psi(x)\Psi^*(x) \\ &= A^2\int_{-\infty}^{\infty} dx\ xexp(-\alpha^2x^2) \end{aligned}$$

因为$xexp(-\alpha^2x^2)$是奇函数,所以$<x>=0$.
(2) 计算归一化因子A

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} dx|\Psi(x)|^2 \\ &= A^2\int_{-\infty}^{\infty} dx\ exp(-\alpha^2x^2) \\ &= A^2\frac{\sqrt{\pi}}{\alpha} \\ A &= (\frac{\alpha^2}{\pi})^{\frac{1}{4}} \end{aligned}$$

$$\begin{aligned} \varphi(p) &= \frac{1}{(2\pi\hbar)^{\frac{1}{2}}}A\int_{-\infty}^{\infty} dxexp(-\alpha^2x^2/2+i\frac{p_0x}{\hbar})\cdot exp(-i\frac{px}{\hbar}) \\ &= \frac{1}{(2\pi\hbar)^{\frac{1}{2}}}A\int_{-\infty}^{\infty} dxexp\{-\alpha^2[x^2-\frac{2i(p_0-p)x}{\alpha^2\hbar}]\} \\ &= \frac{1}{(2\pi\hbar)^{\frac{1}{2}}}A\int_{-\infty}^{\infty} dxexp[-\frac{\alpha^2}{2}(x-\frac{i(p_0-p)}{\alpha^2\hbar})^2]\cdot exp(-\frac{(p-p_0)^2}{2\alpha^2\hbar^2}) \\ &= \frac{1}{(\pi\hbar^2\alpha^2)^{\frac{1}{4}}}exp[-\frac{(p-p_0)^2}{2\alpha^2\hbar^2}] \end{aligned}$$

(3) 令$p\prime=p-p_0$, $\varphi\prime(p)=\frac{1}{(\pi\hbar^2\alpha^2)^{\frac{1}{4}}}exp[-\frac{p^2}{2\alpha^2\hbar^2}]$

$$\begin{aligned} <p> &= \int_{-\infty}^{\infty} dp\ p|\varphi(p)|^2 \\ &= \int_{-\infty}^{\infty} dp\prime(p_0+p\prime)|\varphi\prime(p\prime)|^2 \\ &= \int_{-\infty}^{\infty} dp\prime\ p_0|\varphi\prime(p\prime)|^2 \\ &= p_0 \end{aligned}$$

$$\begin{aligned} <p^2> &= \int_{-\infty}^{\infty} dp\ p^2|\varphi(p)|^2 \\ &= \int_{-\infty}^{\infty} dp\prime(p_0+p\prime)^2|\varphi\prime(p\prime)|^2 \end{aligned}$$

$$
\begin{aligned}
&= \int_{-\infty}^{\infty} dp\prime\ (p_0^2 + 2p_0 p\prime + p\prime^2)|\varphi\prime(p\prime)|^2 \\
&= p_0^2 + \int_{-\infty}^{\infty} dp\prime\ (p\prime)^2|\varphi\prime(p\prime)|^2 \\
&= p_0^2 + \frac{\alpha^2\hbar^2}{2}
\end{aligned}
$$

其中利用了公式$\int_{-\infty}^{\infty} dx x^{2n} e^{-x^2} = \frac{(2n-1)!!}{2^n}\sqrt{\pi}$.

$$\Delta p = (< p^2 > - < p >^2)^{\frac{1}{2}} = \frac{\alpha\hbar}{\sqrt{2}}$$

(4) $< T >= \frac{<p^2>}{2m} = \frac{p_0^2}{2m} + \frac{\alpha^2\hbar^2}{4m}$

3 推导出动量表象下的坐标算符

$$\langle x\rangle = \int |\psi(x)|^2 x dx = \int \psi^*(x)\psi(x) x dx$$

$$\psi^*(x) = \int \frac{1}{(2\pi\hbar)^{\frac{1}{2}}}\varphi^*(p) e^{-i\frac{p}{\hbar}x} dp$$

$$\Rightarrow \langle x\rangle = \int \frac{dpdx}{(2\pi\hbar)^{\frac{1}{2}}}\varphi^*(p) e^{-i\frac{p}{\hbar}x} x\psi(x)$$

$$i\hbar\frac{d}{dp} e^{-i\frac{p}{\hbar}x} = x e^{-i\frac{p}{\hbar}x}$$

$$\Rightarrow \langle x\rangle = \int \frac{dpdx}{(2\pi\hbar)^{\frac{1}{2}}}\varphi^*(p) i\hbar\frac{d}{dp} e^{-i\frac{p}{\hbar}x}\psi(x)$$

$$\varphi(p) = \frac{1}{(2\pi\hbar)^{\frac{1}{2}}}\int \psi(x) e^{-i\frac{p}{\hbar}x} dx$$

$$\Rightarrow \langle x\rangle = \int \varphi^*(p)(i\hbar\frac{d}{dp})\varphi(p) dp$$

$$\Rightarrow \widehat{x} = i\hbar\frac{d}{dp}$$

4. 一张纸画满间距为l的平行线. 往纸上扔长度为l的火柴. 问火柴跟线相交的几率是多少? (提示: 假设扔下的火柴与线成θ角,其中心距离最近的线为x,那么它跟线相交的条件是什么?)

解: 设火柴与垂直方向成θ角,竖直距离为$lcos\theta$. 给定角度θ,火柴与线相交的条件为其中心距线的距离$x < \frac{lcos\theta}{2}$. 我们知道火柴中心距离最近的线的距离按$\frac{1}{l}$的几率密度分布在$[-\frac{l}{2}, \frac{l}{2}]$区间,因此,在给定$\theta$下,火柴与线相交的几率为$\frac{2lcos\theta}{2l} = cos\theta$ (θ的取值范围为$(-\frac{\pi}{2}, \frac{\pi}{2}]$), θ在$-\frac{\pi}{2}$到$\frac{\pi}{2}$之间等几率密度取值,即$\rho(\theta) = \frac{1}{\pi}$. 因此, 角度为$\theta$同时与平行线相交的几率为$\rho(\theta)d\theta\cdot cos\theta$. 则不论角度如何,与平行线相交的几率为

$$\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \rho(\theta)d\theta\cdot cos\theta = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \frac{cos\theta}{\pi}d\theta = \frac{2}{\pi}$$