Homework4 答案

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1 Hermite多项式满足递推关系

$$H_{n+1}(\xi) - 2\xi H_n(\xi) + 2nH_{n-1}(\xi) = 0 \tag{1}$$

证明：

$$x\psi_n(x) = \frac{1}{\alpha}[\sqrt{\frac{n}{2}}\psi_{n-1} + \sqrt{\frac{n+1}{2}}\psi_{n+1}(x)] \tag{2}$$

和

$$x^2\psi_n(x) = \frac{1}{2\alpha^2}[\sqrt{n(n-1)}]\psi_{n-2} + (2n+1)\psi_n(x) + \sqrt{(n+1)(n+2)}\psi_{n+2}(x)] \tag{3}$$

并求出$\psi_n(x)$态下$< x >, < V >$.

解：已知

$$\begin{aligned}\psi_n(x) &= N_n exp(-\frac{1}{2}\alpha^2 x^2)H_n(\alpha x) \\ &= N_n exp(-\frac{1}{2}\xi^2)H_n(\xi)\end{aligned}$$

其中$N_n = (\frac{\alpha}{\sqrt{\pi}2^n n!})^{\frac{1}{2}}$，$\xi = \alpha x$，$\alpha = \sqrt{\frac{m\omega}{\hbar}}$,将上式代入$\alpha x\psi_n(x)$得

$$\begin{aligned}\alpha x\psi_n(x) &= \xi \cdot N_n exp(-\frac{1}{2}\xi^2)H_n(\xi) \\ &= N_n exp(-\frac{1}{2}\xi^2)\xi H_n(\xi)\end{aligned}$$

利用递推关系，可以进一步得到

$$\begin{aligned}\alpha x\psi_n(x) &= N_n exp(-\frac{1}{2}\xi^2)[\frac{1}{2}H_{n+1}(\xi) + nH_{n-1}(\xi)] \\ &= \sqrt{\frac{n+1}{2}}N_{n+1}exp(-\frac{1}{2}\xi^2)H_{n+1}(\xi) + \sqrt{\frac{n}{2}}N_{n-1}exp(-\frac{1}{2}\xi^2)H_{n-1}(\xi) \\ &= \sqrt{\frac{n+1}{2}}\psi_{n+1}(x) + \sqrt{\frac{n}{2}}\psi_{n-1}(x)\end{aligned}$$

所以

$$x\psi_n(x) = \frac{1}{\alpha}[\sqrt{\frac{n}{2}}\psi_{n-1}(x) + \sqrt{\frac{n+1}{2}}\psi_{n+1}(x)]$$

利用上式迭代可得

$$\begin{aligned}
\alpha^2 x^2 \psi_n(x) &= \alpha x \cdot [\sqrt{\frac{n+1}{2}}\psi_{n+1}(x) + \sqrt{\frac{n}{2}}\psi_{n-1}(x)] \\
&= \sqrt{\frac{n+1}{2}}\alpha x\psi_{n+1}(x) + \sqrt{\frac{n}{2}}\alpha x\psi_{n-1}(x) \\
&= \sqrt{\frac{n+1}{2}}[\sqrt{\frac{n+2}{2}}\psi_{n+2}(x) + \sqrt{\frac{n+1}{2}}\psi_n(x)] + \sqrt{\frac{n}{2}}[\sqrt{\frac{n}{2}}\psi_n(x) + \sqrt{\frac{n-1}{2}}\psi_{n-2}(x)] \\
&= \frac{1}{2}[\sqrt{(n+1)(n+2)}\psi_{n+2}(x) + (2n+1)\psi_n(x) + \sqrt{n(n-1)}\psi_{n-2}(x)]
\end{aligned}$$

故

$$x^2\psi_n(x) = \frac{1}{2\alpha^2}[\sqrt{n(n-1)}\psi_{n-2}(x) + (2n+1)\psi_n(x) + \sqrt{(n+1)(n+2)}\psi_{n+2}(x)] \tag{4}$$

计算$< x >_n$，

$$\begin{aligned}
< x >_n &= \int_{-\infty}^{\infty} dx\ \psi_n^*(x) x \psi_n(x) \\
&= \int_{-\infty}^{\infty} dx\ \psi_n^*(x) \cdot \frac{1}{\alpha}[\sqrt{\frac{n}{2}}\psi_{n-1}(x) + \sqrt{\frac{n+1}{2}}\psi_{n+1}(x)]
\end{aligned}$$

利用本征波函数的正交性，可知

$$< x >_n = 0 \tag{5}$$

计算$< V >_n$，

$$\begin{aligned}
< V >_n &= \int_{-\infty}^{\infty} dx\ \psi_n^*(x)\frac{1}{2}m\omega^2 x^2 \psi_n(x) \\
&= \frac{1}{2\alpha^2}\int_{-\infty}^{\infty} dx\ \psi_n(x) \cdot \frac{1}{2}m\omega^2 \cdot [\sqrt{(n+1)(n+2)}\psi_{n+2}(x) + (2n+1)\psi_n(x) + \sqrt{n(n-1)}\psi_{n-2}(x)] \\
&= \frac{2n+1}{4}\frac{m\omega^2}{\alpha^2}\int_{-\infty}^{\infty} dx\ \psi_n(x)\psi_n(x) \\
&= \frac{2n+1}{4}\hbar\omega
\end{aligned}$$

2．利用

$$H_n'(\xi) = 2nH_{n-1}(\xi) \tag{6}$$

类似求出$\frac{d}{dx}\psi_n(x)$，$\frac{d^2}{dx^2}\psi_n(x)$满足的关系，及在$\psi_n(x)$态，动量和动能的期望值。

解：同上可得

$$\begin{aligned}
\frac{d}{dx}\psi_n(x) &= N_n\frac{d}{dx}[exp(-\frac{1}{2}\alpha^2x^2)H_n(\alpha x)] \\
&= \alpha N_n\frac{d}{d\xi}[exp(-\frac{1}{2}\xi^2)H_n(\xi)] \\
&= \alpha N_n(-\xi)exp(-\frac{1}{2}\xi^2)H_n(\xi)+\alpha N_n exp(-\frac{1}{2}\xi^2)H_n'(\xi) \\
&= \alpha N_n(-\xi)exp(-\frac{1}{2}\xi^2)H_n(\xi)+\alpha N_n exp(-\frac{1}{2}\xi^2)2nH_{n-1}(\xi) \\
&= \alpha N_n exp(-\frac{1}{2}\xi^2)[2nH_{n-1}(\xi)-\xi H_n(\xi)] \\
&= \alpha N_n exp(-\frac{1}{2}\xi^2)[nH_{n-1}(\xi)-\frac{1}{2}H_{n+1}(\xi)] \\
&= \alpha[\sqrt{\frac{n}{2}}\psi_{n-1}(x)-\sqrt{\frac{n+1}{2}}\psi_{n+1}(x)]
\end{aligned}$$

$$\begin{aligned}
\frac{d^2}{dx^2}\psi_n(x) &= \alpha\frac{d}{dx}[\sqrt{\frac{n}{2}}\psi_{n-1}(x)-\sqrt{\frac{n+1}{2}}\psi_{n+1}(x)] \\
&= \alpha\sqrt{\frac{n}{2}}\cdot\alpha[\sqrt{\frac{n-1}{2}}\psi_{n-2}(x)-\sqrt{\frac{n}{2}}\psi_n(x)]- \\
&\quad \alpha\sqrt{\frac{n+1}{2}}\cdot\alpha[\sqrt{\frac{n+1}{2}}\psi_n(x)-\sqrt{\frac{n+2}{2}}\psi_{n+2}(x)] \\
&= \frac{\alpha^2}{2}[\sqrt{n(n-1)}\psi_{n-2}-(2n+1)\psi_n(x)+\sqrt{(n+1)(n+2)}\psi_{n+2}(x)]
\end{aligned}$$

计算$<p>_n$，

$$\begin{aligned}
<p>_n &= -i\hbar<\frac{d}{dx}>_n \\
&= -i\hbar\int_{-\infty}^{\infty}dx\ \psi_n^*(x)\frac{d}{dx}\psi_n(x) \\
&= -i\hbar\alpha\int_{-\infty}^{\infty}dx\ \psi_n^*(x)[\sqrt{\frac{n}{2}}\psi_{n-1}(x)-\sqrt{\frac{n+1}{2}}\psi_{n+1}(x)] \\
&= 0
\end{aligned}$$

计算$< \frac{p^2}{2m} >_n$,

$$
\begin{aligned}
< \frac{p^2}{2m} >_n &= -\frac{\hbar^2}{2m} < \frac{d^2}{dx^2} >_n \\
&= -\frac{\hbar^2}{2m}\int_{-\infty}^{\infty} dx\ \psi_n^*(x)\frac{d^2}{dx^2}\psi_n(x) \\
&= -\frac{\hbar^2\alpha^2}{4m}\int_{-\infty}^{\infty} dx\ \psi_n^*(x)[\sqrt{n(n-1)}\psi_{n-2}(x) - (2n+1)\psi_n(x) + \sqrt{(n+2)(n+2)}\psi_{n+2}(x)] \\
&= \frac{\hbar^2\alpha^2}{4m}\int_{-\infty}^{\infty} dx\ \psi_n^*(x)(2n+1)\psi_n(x) \\
&= \frac{(2n+1)\hbar^2\alpha^2}{4m} \\
&= \frac{2n+1}{4}\hbar\omega
\end{aligned}
$$

3．如果谐振子处于第n个定态，计算不确定度Δx，$\Delta p, \Delta x \cdot \Delta p$.

解：利用题1和题2结果计算可得

$$
\begin{aligned}
< x >_n &= 0 \\
< x^2 >_n &= \frac{2n+1}{2\alpha^2} \\
< p >_n &= 0 \\
< p^2 >_n &= < (-i\hbar\frac{d}{dx})^2 >_n = -\hbar^2 < \frac{d^2}{dx^2} >_n = \hbar^2\frac{\alpha^2}{2}(2n+1) = \frac{\hbar^2\alpha^2}{2}(2n+1) \\
\Delta x &= (< x^2 >_n - < x >_n^2)^{\frac{1}{2}} \\
&= \frac{\sqrt{\frac{2n+1}{2}}}{\alpha} \\
\Delta p &= (< p^2 >_n - < p >_n^2)^{\frac{1}{2}} \\
&= \sqrt{\frac{2n+1}{2}}\hbar\alpha \\
\Delta x \cdot \Delta p &= \frac{(2n+1)\hbar}{2} \geq \frac{\hbar}{2}
\end{aligned}
$$

4．半谐振子$V(x) = \frac{1}{2}m\omega^2x^2$,当$x > 0$; $V(x) = \infty$,当$x < 0$.求允许的能级。
（提示：无需太多计算）
解：因为$x > 0$区域势能为无穷大，所以$\psi_n(x)$为零。$x < 0$区域为谐振子势场，波函数与全谐

振子波函数相同，为：

$$\psi_n(x) = N_n exp(-\frac{1}{2}\alpha^2 x^2)H_n(\alpha x) = N_n exp(-\frac{1}{2}\xi^2)H_n(\xi) \quad (7)$$

又由波函数连续性条件：在$x=0$处，$\psi_n(x)=0$。其中厄米多项式在n取奇数时在$x=0$处为零,在n取偶数时不满足在$x=0$处为零。故n只能取奇数，则允许的能级为:

$$E=(2n+3/2)\hbar\omega, \quad n=0,1,2... \quad (8)$$

5．粒子以动量$\hbar k$从左边入射，遇到势场

$$V(x)=\begin{cases} 0, & x<0 \\ V_0>0, & x>0 \end{cases}$$

(a)求反射系数和透射系数。($E>V_0$，$E<V_0$分别讨论)
(b)如果$V_0<0$,上诉势能可以描述一个自由中子进入原子核：从$V=0$到内部$V_0=-12Mev$.假设一个由裂变产生的中子动能为$4Mev$,轰击原子核。请问它被吸收的几率多大？能否触发新的裂变？（提示：$T=1-R$是穿透的几率）

解：由Schrodinger方程

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x)=E\psi(x),\ x<0 \quad (9)$$

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x)+V_0\psi(x)=E\psi(x),\ x>0 \quad (10)$$

解得

$$\psi(x)=\begin{cases} e^{ikx}+re^{-ikx},\ x<0 & (11) \\ se^{ik'x},\ x\geq 0 & (12) \end{cases}$$

其中

$$k = \sqrt{2mE/\hbar} \quad (13)$$

$$k' = \sqrt{2m(E-V_0)/\hbar} \quad (14)$$

由式（11）、（12）得，入射、反射、透射粒子流密度分别为

$$j_i = \frac{\hbar k}{m} \quad (15)$$

$$j_r = -|r|^2 j_i \quad (16)$$

$$j_s = \frac{k'}{k}|s|^2 j_i \quad (17)$$

反射系数，透射系数分别为

$$R = |\frac{j_r}{j_i}| = |r|^2 \tag{18}$$

$$S = |\frac{j_s}{j_i}| = Re(\frac{k'}{k})|s|^2 \tag{19}$$

x=0点波函数及导数的连续性给出

$$1 + r = s \tag{20}$$

$$1 - r = \frac{k'}{k}s \tag{21}$$

解得

$$r = \frac{k-k'}{k+k'} = \frac{\sqrt{E}-\sqrt{E-V_0}}{\sqrt{E}+\sqrt{E-V_0}} \tag{22}$$

$$s = \frac{2k}{k+k'} = \frac{2\sqrt{E}}{\sqrt{E}+\sqrt{E-V_0}} \tag{23}$$

当$E > V_0$时, s,r为实数，

$$R = (\frac{\sqrt{E}-\sqrt{E-V_0}}{\sqrt{E}+\sqrt{E-V_0}})^2 \tag{24}$$

$$S = 1 - R = 1 - (\frac{\sqrt{E}-\sqrt{E-V_0}}{\sqrt{E}+\sqrt{E-V_0}})^2 \tag{25}$$

当$E < V_0$时,$\sqrt{E-V_0} = i\sqrt{V_0-E}$为虚数，

$$\begin{aligned} R &= |\frac{\sqrt{E}-i\sqrt{V_0-E}}{\sqrt{E}+i\sqrt{V_0-E}}|^2 \\ &= \frac{\sqrt{E}-i\sqrt{V_0-E}}{\sqrt{E}+i\sqrt{V_0-E}} \cdot \frac{\sqrt{E}+i\sqrt{V_0-E}}{\sqrt{E}-i\sqrt{V_0-E}} \\ &= 1 \end{aligned} \tag{26}$$

$$S = 1 - R = 0 \tag{27}$$

(b)中子在核中$E = 4Mev, V_0 = -12Mev$,带入（25）式，得:

$$\begin{aligned} S &= 1 - R = 1 - (\frac{\sqrt{4}-\sqrt{4+12}}{\sqrt{4}+\sqrt{4+12}})^2 \\ &= 1 - (\frac{2}{6})^2 \\ &= \frac{8}{9} \end{aligned}$$

中子的透射概率即为中子被吸收的概率,为$\frac{8}{9}$.

中子在核中$E = 4Mev, V_0 = -12Mev$,相当于取$V_0 = 0, x < 0; V_0 = 12MeV; x > 0; E = 16MeV$带入（25）式，得:

$$\begin{aligned} S &= 1 - R = 1 - (\frac{\sqrt{16}-\sqrt{4}}{\sqrt{16}+\sqrt{4}})^2 \\ &= \frac{8}{9} \end{aligned}$$

中子从原子核透射出去概率即为触发新裂变的概率。所以中子进入原子核后会触发新裂变。

6 (1)考虑一个经典谐振子，位移$x = Acos(wt + \delta)$,如果我们不知道它的初始位相δ,或者说它的初始位相位于0到2π,请问它处于x附近的机率密度。(等效于大量相同能量谐振子，位相随机，在x附近发现它们的几率).(提示：Griffith书example1.1)
2)写一个程序，画出量子谐振子的$|\psi_{100}(x)|^2$,计算一定dx内的平均几率密度，与上面的经典几率密度做对比(A由E_n决定)
1)振子位相处于$(0, 2\pi)$中任意δ附近的几率为$\frac{1}{2\pi}\delta$
δ对应的dx为：$dx = -A\sin(\omega t + \delta)d\delta$
振子位于$(x, x + dx)$的几率为：

$$p(x)dx = \frac{dx}{2\pi A(1-\cos^2(\omega t+\delta))^{\frac{1}{2}}} = \frac{dx}{2\pi\sqrt{A^2-x^2}}$$

当δ从0变化到2π时候，x两次通过同一位置

$$\therefore p(x)dx = \frac{dx}{\pi\sqrt{A^2-x^2}}$$

此几率分布对任意时刻都成立，纯粹由于位相不确定造成。
2)

$$\begin{aligned} E_{classical} &= \frac{1}{2}m\omega^2A^2 \quad E^n_{quantum} = (n+\frac{1}{2})\hbar\omega \\ E_{classical} &= E^n_{quantum} \Rightarrow A = \sqrt{(2n+1)\frac{\hbar}{m\omega}} \\ P_{classical}(x) &= \sqrt{\frac{m\omega}{\hbar}}\frac{1}{\pi\sqrt{(2n+1)-\xi^2}} \\ P_{quantum}(x) &= |\psi_n(x)|^2 = \sqrt{\frac{m\omega}{\pi\hbar}}\frac{1}{2^n n!}H^*_n(\xi)H_n(\xi)\exp(-\xi^2), \xi = \sqrt{\frac{m\omega}{\hbar}}x \end{aligned}$$

7 Griffith 2.38.一个质量为m的粒子处于无限深势阱的基态，突然将势阱宽度增加到原来的两倍-将势阱右端从a位置移动到$2a$处，保持波函数未被扰动，现在测量其能量。
a)最可能的的测量结果及其概率
b)第二可能的结果及其概率

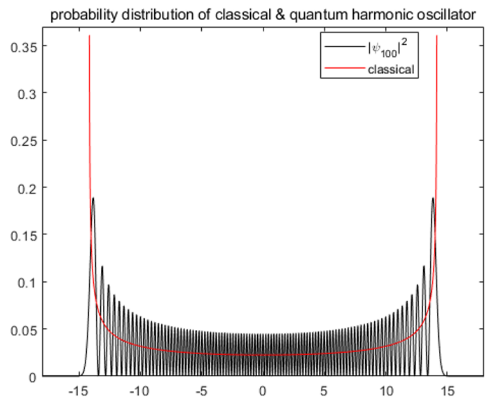

Figure 1: probability density of quantum and classical osillator

c)能量的期望值

$$(a)\text{新的能级：}\ E_n = \frac{n^2\pi^2\hbar^2}{2m(2a)^2};\quad \Psi(x,0)\sqrt{\frac{2}{a}}\sin(\frac{\pi}{a}x), \psi_n(x) = \sqrt{\frac{2}{2a}}\sin(\frac{n\pi}{2a}x).$$

$$\begin{aligned}
c_n &= \frac{\sqrt{2}}{a}\int_0^a \sin(\frac{\pi}{a}x)\sin(\frac{2\pi}{a}x)dx = \frac{\sqrt{2}}{2a}\int_0^a \left\{\cos\left[(\frac{n}{2}-1)\frac{\pi x}{a}\right] - \cos\left[(\frac{n}{2}+1)\frac{\pi x}{a}\right]\right\}dx. \\
&= \frac{1}{\sqrt{2}a}\left\{\frac{\sin\left[(\frac{n}{2}-1)\frac{\pi x}{a}\right]}{(\frac{n}{2}-1)\frac{\pi}{a}} - \frac{\sin\left[(\frac{n}{2}+1)\frac{\pi x}{a}\right]}{(\frac{n}{2}+1)\frac{\pi}{a}}\right\}\Bigg|_0^a \quad n \neq 2 \\
&= \frac{1}{\sqrt{2}\pi}\left\{\frac{\sin\left[(\frac{n}{2}-1)\pi\right]}{(\frac{n}{2}-1)} - \frac{\sin\left[(\frac{n}{2}+1)\pi\right]}{(\frac{n}{2}+1)}\right\} = \frac{\sin\left[(\frac{n}{2}+1)\pi\right]}{\sqrt{2}\pi}\left[\frac{1}{\frac{n}{2}-1} - \frac{1}{\frac{n}{2}+1}\right] \\
&= \frac{4\sqrt{2}}{\pi}\frac{\sin\left[(\frac{n}{2}+1)\pi\right]}{n^2-4} = \begin{cases} 0, & even \\ \pm\frac{4\sqrt{2}}{\pi(n^2-4)}, & odd \end{cases} \\
c_2 &= \frac{\sqrt{2}}{a}\int_0^a \sin^2(\frac{\pi}{a}x)dx = \frac{\sqrt{2}}{a}\int_0^a \frac{1}{2}dx = \frac{1}{\sqrt{2}}.\text{得到}E_n\text{的概率为：} \\
P_n &= |c_n|^2 = \begin{cases} \frac{1}{2}, & n=2 \\ \frac{32}{\pi^2(n^-4)^2}, & n\ is\ odd \\ 0. & otherwise \end{cases}
\end{aligned}$$

最可能$E_2 = \frac{\pi^2\hbar^2}{2ma^2}$, 概率$P_2 = \frac{1}{2}$, $E_1 = \frac{\pi^2\hbar^2}{8ma^2}$, 概率$P_1 = \frac{32}{9\pi^2} = 0,36025$
$\langle H \rangle = \int \Psi^* H \Psi dx = \frac{2}{a}\int_0^a \sin(\frac{\pi}{a}x)(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2})\sin(\frac{\pi}{a}x)dx$可以看出平均值与移动前相同：$\frac{\pi^2\hbar^2}{2ma^2}$.

8 Griffiths2.39(a)无限深势阱中粒子的波函数在量子返回时间$T=\frac{4ma^2}{\pi\hbar}$后会回到初始的状态，即$\Psi(x,T)=\Psi(x,0)$
(b)经典情况下的返回时间是多少,对一个能量为E的粒子?
(c)能量为多少时两个时间相等?
无限深势阱含时薛定谔方程通解为:

$$\Psi(x,t)=\sum_{n=1}^{\infty}c_n\psi_n(x)e^{-i(n^2\pi^2\hbar/2ma^2)t}$$

$$当\ \frac{n^2\pi^2\hbar}{2ma^2}T=\frac{n^2\pi^2\hbar}{2ma^2}\frac{4ma^2}{\pi\hbar}=2\pi n^2,\Rightarrow e^{-i(n^2\pi^2\hbar/2ma^2)(t+T)}=e^{-i(n^2\pi^2\hbar/2ma^2)t}e^{-i2\pi n^2}$$

$$\because n^2是整数,e^{-i2\pi n^2}=1.\therefore\Psi(x,t+T)=\Psi(x,t).QED$$

(b)经典的往返时间$T_c=2a/v$,速度由:

$$E=\frac{1}{2}mv^2\Rightarrow v=\sqrt{\frac{2E}{m}}\Rightarrow T_c=a\sqrt{\frac{2m}{E}}$$

(c)二者时间相等时:

$$\frac{4ma^2}{\pi\hbar}=a\sqrt{\frac{2E}{m}}\Rightarrow E=\frac{\pi^2\hbar^2}{8ma^2}=\frac{E_1}{4}$$