Homework6 答案

November 8, 2019

1.在S_z表象下，求解S_y的本征方程，写出本征值和本征矢。
解：在S_z表象下，

$$S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{1}$$

设本征矢为

$$|\alpha >= \begin{pmatrix} a \\ b \end{pmatrix} \tag{2}$$

对应的本征值为λ,则有:

$$\frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \end{pmatrix} \tag{3}$$

从而得到久期方程

$$\begin{vmatrix} -\lambda & -\frac{\hbar}{2}i \\ \frac{\hbar}{2}i & -\lambda \end{vmatrix} = 0 \tag{4}$$

解得$\lambda = \pm\frac{\hbar}{2}$, 将本征值依次带入式（3）得到相应的归一化本征矢

$$\lambda_1 = \frac{\hbar}{2}, \quad |+, S_y > \;=\; \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} \tag{5}$$

$$\lambda_2 = -\frac{\hbar}{2}, \quad |-, S_y > \;=\; \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix} \tag{6}$$

2. 一个自旋1/2的粒子处于$|+, z >$态，计算$< S_y >, < S_x >$。（请使用两种方式计算：1. 根据量子力学测量原理，同时回答测量值与相应几率；2.利用期望值的矩阵计算公式）

解：(1)根据量子力学测量原理，

$$|+, z >= \frac{1}{\sqrt{2}}(|+, x > + |-, x >) \tag{7}$$

所以测得S_x为$\frac{\hbar}{2}$的概率为$\frac{1}{2}$,为$\frac{-\hbar}{2}$的概率为$\frac{1}{2}$。

$$|+,z>=\frac{1}{\sqrt{2}}(|+,y>+|-,y>) \tag{8}$$

所以测得S_y为$\frac{\hbar}{2}$的概率为$\frac{1}{2}$,为$\frac{-\hbar}{2}$的概率为$\frac{1}{2}$。
综上,

$$<S_x>=\frac{\hbar}{2}*\frac{1}{2}+(-\frac{\hbar}{2})*\frac{1}{2}=0 \tag{9}$$
$$<S_y>=\frac{\hbar}{2}*\frac{1}{2}+(-\frac{\hbar}{2})*\frac{1}{2}=0 \tag{10}$$

(2)利用期望值的矩阵计算公式:
在S_z表象下,初始态为:

$$|+,S_z>=\begin{pmatrix}1\\0\end{pmatrix} \tag{11}$$

$$\begin{aligned}<S_y> &= \frac{\hbar}{2}\begin{pmatrix}1,0\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\\ &= 0\end{aligned} \tag{12}$$

$$\begin{aligned}<S_x> &= \frac{\hbar}{2}\begin{pmatrix}1,0\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\\ &= 0\end{aligned} \tag{13}$$

3. $\vec{n}$为任意方向(θ,ϕ)的单位矢量,$\sigma_n=\vec{n}\cdot\vec{\sigma}$为泡得算符在该方向的投影。(也就是Stern-Galach装置沿$\vec{n}$方向设置,观察该方向的偏转)
(1)写出S_z表象下,σ_n的矩阵表示,并求出本征值和本征矢;
(2)对于σ_n的本征矢计算$\sigma_x,\sigma_y,\sigma_z$的期望值和标准偏差;
(3)如果处于σ_n的+1的本征态的一束中性粒子自旋1/2粒子通过沿x方向设置的Stern-Galach装置,会分裂成几束?强度比是多少?(强度比就是粒子数比);
(4)如果中性1/2粒子通过x方向设置的Stern-Galach装置后,取$S_x=\frac{\hbar}{2}$的一束,让它通过沿$\vec{n}$方向设置的Stern-Galach装置,请问分成几束?强度比是多少?

解:(1)$\sigma_x,\sigma_y,\sigma_z$在$S_z$表象下的矩阵表示为

$$\sigma_x=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\ \sigma_y=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},\ \sigma_z=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},$$

代入$\sigma_n = \vec{n} \cdot \vec{\sigma}$得到

$$\begin{aligned}
\sigma_n &= n_x \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + n_y \cdot \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + n_z \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
&= cos\phi sin\theta \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + sin\phi sin\theta \cdot \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + cos\theta \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
&= \begin{pmatrix} cos\theta & cos\phi sin\theta - isin\phi sin\theta \\ cos\phi sin\theta + isin\phi sin\theta & -cos\theta \end{pmatrix} \\
&= \begin{pmatrix} cos\theta & sin\theta e^{-i\phi} \\ sin\theta e^{i\phi} & -cos\theta \end{pmatrix} \qquad (14)
\end{aligned}$$

设σ_n的本征值和本征矢分别为$\lambda, |\alpha >= \binom{a}{b}$，且有$|a|^2 + |b|^2 = 1$, 由shrodinger方程可得

$$\begin{pmatrix} cos\theta & sin\theta e^{-i\phi} \\ sin\theta e^{i\phi} & -cos\theta \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \end{pmatrix} \qquad (15)$$

从而得到久期方程

$$\begin{vmatrix} cos\theta - \lambda & sin\theta e^{-i\phi} \\ sin\theta e^{i\phi} & -cos\theta - \lambda \end{vmatrix} = 0 \qquad (16)$$

解得$\lambda = \pm 1$, 将本征值依次带入式（15）得到相应的归一化本征矢:

$$\lambda_1 = 1, \quad |+, \sigma_n > \;=\; \begin{pmatrix} cos\frac{\theta}{2} e^{-i\frac{\phi}{2}} \\ sin\frac{\theta}{2} e^{i\frac{\phi}{2}} \end{pmatrix} \qquad (17)$$

$$\lambda_2 = -1, \quad |-, \sigma_n > \;=\; \begin{pmatrix} sin\frac{\theta}{2} e^{-i\frac{\phi}{2}} \\ -cos\frac{\theta}{2} e^{i\frac{\phi}{2}} \end{pmatrix} \qquad (18)$$

以上本征矢有规范自由度，可以乘以任意模为一的复数，比如$\exp i\phi$

(2) σ_n的本征值为1时：$\sigma_x, \sigma_y, \sigma_z$ 的期望值为

$$\begin{aligned}
< \sigma_x > &= \begin{pmatrix} cos\frac{\theta}{2} e^{i\frac{\phi}{2}}, & sin\frac{\theta}{2} e^{-i\frac{\phi}{2}} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} cos\frac{\theta}{2} e^{-i\frac{\phi}{2}} \\ sin\frac{\theta}{2} e^{i\frac{\phi}{2}} \end{pmatrix} \\
&= sin\theta cos\phi \qquad (19) \\
< \sigma_y > &= \begin{pmatrix} cos\frac{\theta}{2} e^{i\frac{\phi}{2}}, & sin\frac{\theta}{2} e^{-i\frac{\phi}{2}} \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} cos\frac{\theta}{2} e^{-i\frac{\phi}{2}} \\ sin\frac{\theta}{2} e^{i\frac{\phi}{2}} \end{pmatrix} \\
&= sin\theta sin\phi \qquad (20) \\
< \sigma_z > &= \begin{pmatrix} cos\frac{\theta}{2} e^{i\frac{\phi}{2}}, & sin\frac{\theta}{2} e^{-i\frac{\phi}{2}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} cos\frac{\theta}{2} e^{-i\frac{\phi}{2}} \\ sin\frac{\theta}{2} e^{i\frac{\phi}{2}} \end{pmatrix} \\
&= cos\theta \qquad (21)
\end{aligned}$$

σ_n的本征值为-1时：$\sigma_x, \sigma_y, \sigma_z$ 的期望值为

$$\begin{aligned} <\sigma_x> &= \begin{pmatrix} sin\frac{\theta}{2}e^{i\frac{\phi}{2}}, & -cos\frac{\theta}{2}e^{-i\frac{\phi}{2}} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} sin\frac{\theta}{2}e^{-i\frac{\phi}{2}} \\ -cos\frac{\theta}{2}e^{i\frac{\phi}{2}} \end{pmatrix} \\ &= -sin\theta cos\phi \end{aligned} \tag{22}$$

$$\begin{aligned} <\sigma_y> &= \begin{pmatrix} sin\frac{\theta}{2}e^{i\frac{\phi}{2}}, & -cos\frac{\theta}{2}e^{-i\frac{\phi}{2}} \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} sin\frac{\theta}{2}e^{-i\frac{\phi}{2}} \\ -cos\frac{\theta}{2}e^{i\frac{\phi}{2}} \end{pmatrix} \\ &= -sin\theta sin\phi \end{aligned} \tag{23}$$

$$\begin{aligned} <\sigma_z> &= \begin{pmatrix} sin\frac{\theta}{2}e^{i\frac{\phi}{2}}, & -cos\frac{\theta}{2}e^{-i\frac{\phi}{2}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} sin\frac{\theta}{2}e^{-i\frac{\phi}{2}} \\ -cos\frac{\theta}{2}e^{i\frac{\phi}{2}} \end{pmatrix} \\ &= -cos\theta \end{aligned} \tag{24}$$

由于

$$\sigma_x^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_{2\times 2} \tag{25}$$

$$\sigma_y^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_{2\times 2} \tag{26}$$

$$\sigma_z^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_{2\times 2} \tag{27}$$

故

$$<\sigma_x^2>= 1 \tag{28}$$

$$<\sigma_y^2>= 1 \tag{29}$$

$$<\sigma_z^2>= 1 \tag{30}$$

σ_n的本征值为1时,$\sigma_x, \sigma_y, \sigma_z$的标准偏差为

$$\begin{aligned} \Delta\sigma_x &= \sqrt{<\sigma_x^2> - <\sigma_x>^2} = \sqrt{1 - sin^2\theta cos^2\phi} \\ \Delta\sigma_y &= \sqrt{<\sigma_y^2> - <\sigma_y>^2} = \sqrt{1 - sin^2\theta sin^2\phi} \\ \Delta\sigma_z &= \sqrt{<\sigma_z^2> - <\sigma_z>^2} = |sin\theta| \end{aligned}$$

σ_n的本征值为-1时,$\sigma_x, \sigma_y, \sigma_z$的标准偏差为

$$\begin{aligned} \Delta\sigma_x &= \sqrt{<\sigma_x^2> - <\sigma_x>^2} = \sqrt{1 - sin^2\theta cos^2\phi} \\ \Delta\sigma_y &= \sqrt{<\sigma_y^2> - <\sigma_y>^2} = \sqrt{1 - sin^2\theta sin^2\phi} \\ \Delta\sigma_z &= \sqrt{<\sigma_z^2> - <\sigma_z>^2} = |sin\theta| \end{aligned}$$

(3) 由于σ_x只能取±1两种情况，故一般分成两束。取$\theta = \frac{\pi}{2}$，$\phi = 0$,又易求σ_x的本征值和本征矢为

$$\lambda_1 = 1, \quad |+, S_x> = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{31}$$

$$\lambda_2 = -1, \quad |-, S_x> = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{32}$$

σ_n本征值为1的本征矢$|+,\sigma_n>=\binom{cos\frac{\theta}{2}e^{-i\frac{\phi}{2}}}{sin\frac{\theta}{2}e^{i\frac{\phi}{2}}}$是$|+,S_x>,|-,S_x>$的叠加态:

$$|+,\sigma_n>=a|+,S_x>+b|-,S_x> \tag{33}$$

其中$|a|^2+|b|^2=1$，利用本征态矢正交归一性可得

$$\begin{aligned} a &= <+,S_x|+,\sigma_n> \\ &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} cos\frac{\theta}{2}e^{-i\frac{\phi}{2}} \\ sin\frac{\theta}{2}e^{i\frac{\phi}{2}} \end{pmatrix} \\ &= \frac{1}{\sqrt{2}}(cos\frac{\theta}{2}e^{-i\frac{\phi}{2}}+sin\frac{\theta}{2}e^{i\frac{\phi}{2}}) \end{aligned} \tag{34}$$

$$\begin{aligned} b &= <-,S_x|+,\sigma_n> \\ &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \end{pmatrix}\begin{pmatrix} cos\frac{\theta}{2}e^{-i\frac{\phi}{2}} \\ sin\frac{\theta}{2}e^{i\frac{\phi}{2}} \end{pmatrix} \\ &= \frac{1}{\sqrt{2}}(cos\frac{\theta}{2}e^{-i\frac{\phi}{2}}-sin\frac{\theta}{2}e^{i\frac{\phi}{2}}) \end{aligned} \tag{35}$$

由此得到自旋向上概率$|a|^2$和自旋向下概率$|b|^2$

$$\begin{aligned} |a|^2 &= \frac{1}{\sqrt{2}}(cos\frac{\theta}{2}e^{-i\frac{\phi}{2}}+sin\frac{\theta}{2}e^{i\frac{\phi}{2}})\cdot\frac{1}{\sqrt{2}}(cos\frac{\theta}{2}e^{i\frac{\phi}{2}}+sin\frac{\theta}{2}e^{-i\frac{\phi}{2}}) \\ &= \frac{1}{2}(cos^2\frac{\theta}{2}+sin^2\frac{\theta}{2}+cos\frac{\theta}{2}sin\frac{\theta}{2}(e^{i\phi}+e^{-i\phi})) \\ &= \frac{1}{2}(1+sin\theta cos\phi) \end{aligned} \tag{36}$$

$$|b|^2 = 1-|a|^2=\frac{1}{2}(1-sin\theta cos\phi) \tag{37}$$

因此强度比为

$$\frac{|a|^2}{|b|^2} = \frac{1+sin\theta cos\phi}{1-sin\theta cos\phi} \tag{38}$$

(4)同上，一般分成两束。σ_x本征值为1的本征矢$|+,S_x>=\frac{1}{\sqrt{2}}\binom{1}{1}$是$|+,\sigma_n>=\binom{cos\frac{\theta}{2}e^{-i\frac{\phi}{2}}}{sin\frac{\theta}{2}e^{i\frac{\phi}{2}}}$及$|-,\sigma_n>=\binom{sin\frac{\theta}{2}e^{-i\frac{\phi}{2}}}{-cos\frac{\theta}{2}e^{i\frac{\phi}{2}}}$的叠加态:

$$|+,S_x>=c|+,\sigma_n>+d|-,\sigma_n> \tag{39}$$

其中$|c|^2+|d|^2=1$，利用本征态矢正交归一性可得

$$c = <+,\sigma_n|+,S_x>=<+,S_x|+,\sigma_n>^*=a^* \tag{40}$$

即c为（3）中a的复共轭。因此强度比仍为

$$\frac{|c|^2}{|d|^2} = \frac{1+sin\theta cos\phi}{1-sin\theta cos\phi} \tag{41}$$

4.设某二能级系统的哈密顿量为$H = \epsilon(|1><1| - |2><2| + |1><2| + |2><1|)$ 这里$|1>, |2>$是正交归一完备基。求H的本征值和归一化的本征矢。请用H的本征矢写出H的矩阵。

解: 以$|1>, |2>$为基矢，

$$H_{11} =<1|H|1>=<1|\epsilon(|1><1| - |2><2| + |1><2| + |2><1|)|1>= \epsilon \tag{42}$$

其余矩阵元同理可得，则H的矩阵表示为：

$$H = \epsilon \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{43}$$

设H的本征值和本征矢为$E = \epsilon\lambda, |E>= \binom{a}{b}$，由schrodinger方程得

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \lambda\begin{pmatrix} a \\ b \end{pmatrix}$$

$$\sqrt{2}\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \lambda\begin{pmatrix} a \\ b \end{pmatrix} \tag{44}$$

利用题3结果，取$\theta = \frac{\pi}{4}, \phi = 0$得到本征值和本征矢

$$\lambda_1 = \sqrt{2}, E_1 = \sqrt{2}\epsilon, \quad |E_1> = \begin{pmatrix} cos\frac{\pi}{8} \\ sin\frac{\pi}{8} \end{pmatrix} \tag{45}$$

$$\lambda_2 = -\sqrt{2}, E_2 = -\sqrt{2}\epsilon, \quad |E_2> = \begin{pmatrix} sin\frac{\pi}{8} \\ -cos\frac{\pi}{8} \end{pmatrix} \tag{46}$$

以$|E_1>, |E_2>$为基矢，H的矩阵表示为（对角元即为本征值）：

$$H = \sqrt{2}\epsilon\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{47}$$

5.证明：$e^{ia\sigma_z} = cosa + i\sigma_z sina$，其中a是常数。

证明：利用$\sigma_z^2 = 1$

$$\begin{aligned} e^{ia\sigma_z} &= \sum_{n=0}^{\infty}\frac{(ia)^n\sigma_z^n}{n!} \\ &= \sum_{k=0}^{\infty}\frac{(ia)^{2k+1}\sigma_z^{2k+1}}{(2k+1)!} + \sum_{k=0}^{\infty}\frac{(ia)^{2k}\sigma_z^{2k}}{(2k)!} \\ &= i\sigma_z\sum_{k=0}^{\infty}\frac{(-1)^k(a)^{2k+1}}{(2k+1)!} + \sum_{k=0}^{\infty}\frac{(-1)^k(a)^{2k}}{(2k)!} \\ &= cosa + i\sigma_z sina \end{aligned}$$