Homework7 答案

November 11, 2020

1. 均匀磁场中的中性自旋1/2粒子, 磁场方向z, 强度B, $\vec{n}$为任意方向(θ,ϕ)的单位矢量，$\sigma_n = \vec{n}\cdot\vec{\sigma}$为泡利算符在该方向的投影，如果初态是$\sigma_n = 1$的本征态，请解出态的时间演化，并分别计算$S_x, S_y, S_z$的测量值与相应几率。请问要经过多长时间才能回到初态?

解：另一种做法见格里菲斯书例题4.3, 拉莫尔进动。
设哈密顿量为$H = -\mu_s\cdot\vec{B} = \frac{e\hbar}{2m_ec}\vec{\sigma}\cdot\vec{B} = \frac{e\hbar B}{2m_ec}\sigma_z = \hbar\omega\sigma_z$，$\omega = \frac{eB}{2m_ec}$,$\sigma_n = 1$的本征态为

$$|\psi(0)> \quad = \quad \begin{pmatrix} cos\frac{\theta}{2}e^{-i\frac{\phi}{2}} \\ sin\frac{\theta}{2}e^{i\frac{\phi}{2}} \end{pmatrix} \tag{1}$$

哈密顿量相应的本征值和本征矢为

$$E_1 = \hbar\omega, \quad |\psi_1> \quad = \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2}$$

$$E_2 = -\hbar\omega, \quad |\psi_2> \quad = \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3}$$

初始态$|\psi(0)>$是$|\psi_1>, |\psi_2>$的叠加态$|\psi_0> = a|\psi_1> + b|\psi_2>$,其中$|a|^2 + |b|^2 = 1$ 利用本征态矢正交性得

$$\begin{aligned} a &= <\psi_1|\psi(0)> \\ &= \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} cos\frac{\theta}{2}e^{-i\frac{\phi}{2}} \\ sin\frac{\theta}{2}e^{i\frac{\phi}{2}} \end{pmatrix} \\ &= cos\frac{\theta}{2}e^{-i\frac{\phi}{2}} \end{aligned} \tag{4}$$

$$\begin{aligned} b &= <\psi_2|\psi(0)> \\ &= \begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} cos\frac{\theta}{2}e^{-i\frac{\phi}{2}} \\ sin\frac{\theta}{2}e^{i\frac{\phi}{2}} \end{pmatrix} \\ &= sin\frac{\theta}{2}e^{i\frac{\phi}{2}} \end{aligned} \tag{5}$$

得到$|\psi(0)>=cos\frac{\theta}{2}e^{-i\frac{\phi}{2}}|\psi_1>+sin\frac{\theta}{2}e^{i\frac{\phi}{2}}|\psi_2>$, 由此我们可以求出t时刻的波函数为

$$\begin{aligned}|\psi(t)> &= cos\frac{\theta}{2}e^{-i\frac{\phi}{2}}e^{-\frac{i}{\hbar}E_1t}\psi_1+sin\frac{\theta}{2}e^{i\frac{\phi}{2}}e^{-\frac{i}{\hbar}E_2t}\psi_2 \\ &= cos\frac{\theta}{2}e^{-i\frac{\phi}{2}}e^{-i\omega t}\psi_1+sin\frac{\theta}{2}e^{i\frac{\phi}{2}}e^{i\omega t}\psi_2 \\ &= \begin{pmatrix} cos\frac{\theta}{2}e^{-i\frac{\phi}{2}}e^{-i\omega t} \\ sin\frac{\theta}{2}e^{i\frac{\phi}{2}}e^{i\omega t}\end{pmatrix} \end{aligned} \tag{6}$$

利用$e^{ia\sigma_z}=cosa+i\sigma_z sina$，我们也可以直接根据初态得到$|\psi(t)>$:

$$\begin{aligned}|\psi(t)> &= e^{-\frac{i}{\hbar}\hat{H}t}|\psi(0)> \\ &= e^{-i\omega t\sigma_z}|\psi(0)> \\ &= (cos\omega t-i\sigma_z sin\omega t)|\psi(0)> \\ &= \begin{pmatrix} cos\omega t-isin\omega t & 0 \\ 0 & cos\omega t+isin\omega t\end{pmatrix}\begin{pmatrix} cos\frac{\theta}{2}e^{-i\frac{\phi}{2}} \\ sin\frac{\theta}{2}e^{i\frac{\phi}{2}}\end{pmatrix} \\ &= \begin{pmatrix} cos\frac{\theta}{2}e^{-i\frac{\phi}{2}}e^{-i\omega t} \\ sin\frac{\theta}{2}e^{i\frac{\phi}{2}}e^{i\omega t}\end{pmatrix} \end{aligned} \tag{7}$$

计算S_x,S_y,S_z的测量值与相应几率:
方法一：把$|\psi(t)>$按力学量的本征矢展开$|\psi(t)>=a(t)|+>+b(t)|->$,其中$|a|^2+|b|^2=1$
S_x的本征值和本征矢为

$$S_{x+}=\frac{\hbar}{2},\quad |+,S_x> = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \tag{8}$$

$$S_{x-}=-\frac{\hbar}{2},\quad |-,S_x> = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} \tag{9}$$

S_y的本征值和本征矢为

$$S_{y+}=\frac{\hbar}{2},\quad |+,S_y> = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix} \tag{10}$$

$$S_{y-}=-\frac{\hbar}{2},\quad |-,S_y> = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix} \tag{11}$$

S_z的本征值和本征矢为

$$S_{z+}=\frac{\hbar}{2},\quad |+,S_z> = \begin{pmatrix}1\\0\end{pmatrix} \tag{12}$$

$$S_{z-}=-\frac{\hbar}{2},\quad |-,S_z> = \begin{pmatrix}0\\1\end{pmatrix} \tag{13}$$

利用本征态矢正交性得

$$
\begin{aligned}
a_x(t) &= <+, S_x|\psi(t)> \\
&= \frac{1}{\sqrt{2}}\left(\begin{array}{cc} 1, & 1 \end{array}\right)\left(\begin{array}{c} cos\frac{\theta}{2}e^{-i\frac{\phi}{2}}e^{-i\omega t} \\ sin\frac{\theta}{2}e^{i\frac{\phi}{2}}e^{i\omega t} \end{array}\right) \\
&= \frac{1}{\sqrt{2}}(cos\frac{\theta}{2}e^{-i\frac{\phi}{2}}e^{-i\omega t} + sin\frac{\theta}{2}e^{i\frac{\phi}{2}}e^{i\omega t})
\end{aligned} \tag{14}
$$

$$
\begin{aligned}
b_x(t) &= <-, S_x|\psi(t)> \\
&= \frac{1}{\sqrt{2}}\left(\begin{array}{cc} 1, & -1 \end{array}\right)\left(\begin{array}{c} cos\frac{\theta}{2}e^{-i\frac{\phi}{2}}e^{-i\omega t} \\ sin\frac{\theta}{2}e^{i\frac{\phi}{2}}e^{i\omega t} \end{array}\right) \\
&= \frac{1}{\sqrt{2}}(cos\frac{\theta}{2}e^{-i\frac{\phi}{2}}e^{-i\omega t} - sin\frac{\theta}{2}e^{i\frac{\phi}{2}}e^{i\omega t})
\end{aligned} \tag{15}
$$

$$
\begin{aligned}
a_y(t) &= <+, S_y|\psi(t)> \\
&= \frac{1}{\sqrt{2}}\left(\begin{array}{cc} 1, & i \end{array}\right)\left(\begin{array}{c} cos\frac{\theta}{2}e^{-i\frac{\phi}{2}}e^{-i\omega t} \\ sin\frac{\theta}{2}e^{i\frac{\phi}{2}}e^{i\omega t} \end{array}\right) \\
&= \frac{1}{\sqrt{2}}(cos\frac{\theta}{2}e^{-i\frac{\phi}{2}}e^{-i\omega t} + isin\frac{\theta}{2}e^{i\frac{\phi}{2}}e^{i\omega t})
\end{aligned} \tag{16}
$$

$$
\begin{aligned}
b_y(t) &= <-, S_y|\psi(t)> \\
&= \frac{1}{\sqrt{2}}\left(\begin{array}{cc} 1, & -i \end{array}\right)\left(\begin{array}{c} cos\frac{\theta}{2}e^{-i\frac{\phi}{2}}e^{-i\omega t} \\ sin\frac{\theta}{2}e^{i\frac{\phi}{2}}e^{i\omega t} \end{array}\right) \\
&= \frac{1}{\sqrt{2}}(cos\frac{\theta}{2}e^{-i\frac{\phi}{2}}e^{-i\omega t} - isin\frac{\theta}{2}e^{i\frac{\phi}{2}}e^{i\omega t})
\end{aligned} \tag{17}
$$

$$
\begin{aligned}
a_z(t) &= <+, S_z|\psi(t)> \\
&= \left(\begin{array}{cc} 1, & 0 \end{array}\right)\left(\begin{array}{c} cos\frac{\theta}{2}e^{-i\frac{\phi}{2}}e^{-i\omega t} \\ sin\frac{\theta}{2}e^{i\frac{\phi}{2}}e^{i\omega t} \end{array}\right) \\
&= cos\frac{\theta}{2}e^{-i\frac{\phi}{2}}e^{-i\omega t}
\end{aligned} \tag{18}
$$

$$
\begin{aligned}
b_z(t) &= <-, S_z|\psi(t)> \\
&= \left(\begin{array}{cc} 0, & 1 \end{array}\right)\left(\begin{array}{c} cos\frac{\theta}{2}e^{-i\frac{\phi}{2}}e^{-i\omega t} \\ sin\frac{\theta}{2}e^{i\frac{\phi}{2}}e^{i\omega t} \end{array}\right) \\
&= sin\frac{\theta}{2}e^{i\frac{\phi}{2}}e^{i\omega t}
\end{aligned} \tag{19}
$$

得到t时刻S_x, S_y, S_z测量值和相应几率为

$$\begin{aligned}
S_{x+} &= \frac{\hbar}{2}, \quad P_{x+} = |a_x(t)|^2 = \frac{1}{2}[1 + sin\theta(cos\phi cos2\omega t - sin\phi sin2\omega t)] \\
S_{x-} &= -\frac{\hbar}{2}, \quad P_{x-} = |b_x(t)|^2 = \frac{1}{2}[1 - sin\theta(cos\phi cos2\omega t - sin\phi sin2\omega t)] \\
< S_x >_t &= S_{x+}P_{x+} + S_{x-}P_{x-} = \frac{\hbar}{2} sin\theta(cos\phi cos2\omega t - sin\phi sin2\omega t) \qquad (20) \\
S_{y+} &= \frac{\hbar}{2}, \quad P_{y+} = |a_y(t)|^2 = \frac{1}{2}[1 + sin\theta(cos\phi sin2\omega t + sin\phi cos2\omega t)] \\
S_{y-} &= -\frac{\hbar}{2}, \quad P_{y-} = |b_y(t)|^2 = \frac{1}{2}[1 - sin\theta(cos\phi sin2\omega t + sin\phi cos2\omega t)] \\
< S_y >_t &= S_{y+}P_{y+} + S_{y-}P_{y-} = \frac{\hbar}{2} sin\theta(cos\phi sin2\omega t + sin\phi cos2\omega t) \qquad (21) \\
S_{z+} &= \frac{\hbar}{2}, \quad P_{z+} = |a_z(t)|^2 = cos^2\frac{\theta}{2} \\
S_{z-} &= -\frac{\hbar}{2}, \quad P_{z-} = |b_z(t)|^2 = sin^2\frac{\theta}{2} \\
< S_z >_t &= S_{z+}P_{z+} + S_{z-}P_{z-} = \frac{\hbar}{2} cos\theta \qquad (22)
\end{aligned}$$

方法二: 在Heisenberg picture 下计算$< S_x(t) >, < S_y(t) >, < S_z(t) >$.

$$\begin{aligned}
\sigma_x(t) &= e^{\frac{i}{\hbar}Ht}\sigma_x e^{-\frac{i}{\hbar}Ht} \\
&= e^{i\omega t\sigma_z}\sigma_x e^{-i\omega t\sigma_z} \\
&= (cos\omega t + isin\omega t\sigma_z)\sigma_x(cos\omega t - isin\omega t\sigma_z) \\
&= \begin{pmatrix} cos\omega t + isin\omega t & 0 \\ 0 & cos\omega t - isin\omega t \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} cos\omega t - isin\omega t & 0 \\ 0 & cos\omega t + isin\omega t \end{pmatrix} \\
&= \begin{pmatrix} 0 & e^{2i\omega t} \\ e^{-2i\omega t} & 0 \end{pmatrix} \qquad (23) \\
< S_x(t) > &= \frac{\hbar}{2} < \sigma_x(t) > \\
&= \frac{\hbar}{2} < \psi(0)|\sigma_x(t)|\psi(0) > \\
&= \frac{\hbar}{2} sin\theta(cos\phi cos2\omega t - sin\phi sin2\omega t) \qquad (24) \\
\sigma_y(t) &= e^{\frac{i}{\hbar}Ht}\sigma_y e^{-\frac{i}{\hbar}Ht} \\
&= e^{i\omega t\sigma_z}\sigma_y e^{-i\omega t\sigma_z} \\
&= (cos\omega t + isin\omega t\sigma_z)\sigma_y(cos\omega t - isin\omega t\sigma_z) \\
&= \begin{pmatrix} cos\omega t + isin\omega t & 0 \\ 0 & cos\omega t - isin\omega t \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} cos\omega t - isin\omega t & 0 \\ 0 & cos\omega t + isin\omega t \end{pmatrix} \\
&= \begin{pmatrix} 0 & -ie^{2i\omega t} \\ ie^{-2i\omega t} & 0 \end{pmatrix} \qquad (25) \\
< S_y(t) > &= \frac{\hbar}{2} < \sigma_y(t) > \\
&= \frac{\hbar}{2} < \psi(0)|\sigma_y(t)|\psi(0) > \\
&= \frac{\hbar}{2} sin\theta(cos\phi sin2\omega t + sin\phi cos2\omega t) \qquad (26) \\
\sigma_z(t) &= e^{\frac{i}{\hbar}Ht}\sigma_z e^{-\frac{i}{\hbar}Ht} \\
&= e^{i\omega t\sigma_z}\sigma_z e^{-i\omega t\sigma_z} \\
&= (cos\omega t + isin\omega t\sigma_z)\sigma_z(cos\omega t - isin\omega t\sigma_z) \\
&= \begin{pmatrix} cos\omega t + isin\omega t & 0 \\ 0 & cos\omega t - isin\omega t \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} cos\omega t - isin\omega t & 0 \\ 0 & cos\omega t + isin\omega t \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad (27) \\
< S_z(t) > &= \frac{\hbar}{2} < \sigma_z(t) > \\
&= \frac{\hbar}{2} < \psi(0)|\sigma_z(t)|\psi(0) > \\
&= \frac{\hbar}{2} cos\theta \qquad (28)
\end{aligned}$$

现考虑任意一个叠加态，经过有限时间t后，是否一定能够回到原来的状态，如果能，那么经过多久。
为方便讨论，仅考虑在二维希尔伯特空间中的一个叠加态。

$$\begin{aligned} |\psi(0)> &= c_1|\phi_1> + c_2|\phi_2> \quad (29)\\ |\psi(t)> &= c_1|\phi_1> e^{-iE_1t/\hbar} + c_2|\phi_2> e^{-iE_2t/\hbar} \quad (30) \end{aligned}$$

若$|\psi(0)>=|\psi(t)>$则有:

$$\begin{aligned} e^{-iE_1t/\hbar} = 1 \quad (31)\\ e^{-iE_2t/\hbar} = 1 \quad (32) \end{aligned}$$

由式(31)得:

$$E_1t/\hbar = 2n\pi, t = 2n\pi\hbar/E_1 \quad (33)$$

将上式带入(32)式左边，得到$e^{-iE_2t/\hbar} = e^{(-iE_2/\hbar)(2n\pi\hbar/E_1)} = e^{-2n\pi iE_2/E_1}$
若nE_2/E_1为整数，则$e^{-iE_2t/\hbar}$等于1。在本题中$E_2/E_1 = -1$,所以n可取任意整数。将$E_1 = \hbar\omega$带入$t = 2n\pi\hbar/E_1$即得到$t = \frac{2n\pi}{\omega}, (n = 1, 2, 3....)$时，系统回到初始状态。所以系统的周期是$\frac{2\pi}{\omega}$，而$S_x, S_y, S_z$的周期是$\frac{\pi}{\omega}$.

2. 已知Q表象的本征矢为$|q_1>, |q_2>, |q_3>$。哈密顿量矩阵为

$$H = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad (34)$$

（1）求解H的本征方程，给出本征值和本征矢。
（2）

$$|\psi> = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad (35)$$

写出在H表象下的表示。计算$< H >$。（你能用几种方法求？）
（3）如果初态为$|\psi>$，写出之后任意时刻t的态矢量。
（4）写出从Q表象到H表象的变换矩阵。

解：(1) 设H的本征值和本征矢为

$$E = \lambda, |\phi> = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad (36)$$

代入定态Schrodinger方程

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad (37)$$

解得

$$E_1 = 2, \quad |\phi_1 >= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \tag{38}$$

$$E_2 = 1, \quad |\phi_2 >= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \tag{39}$$

$$E_3 = -1, \quad |\phi_3 >= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \tag{40}$$

(2) 把$|\psi>$按H的本征矢展开$|\psi >= c_1|\phi_1 > +c_2|\phi_2 > +c_3|\phi_3 >$,其中$|c_1|^2 + |c_2|^2 + |c_3|^2 = 1$, 利用本征态矢正交性得

$$\begin{aligned} c_1 &= <\phi_1|\psi >= \frac{1}{\sqrt{2}} \\ c_2 &= <\phi_2|\psi >= \frac{1}{2} \\ c_3 &= <\phi_3|\psi >= -\frac{1}{2} \\ |\psi > &= \frac{1}{\sqrt{2}}|\phi_1 > +\frac{1}{2}|\phi_2 > -\frac{1}{2}|\phi_3 > \end{aligned} \tag{41}$$

有两种方法计算$< H >$, 一是在H表象下计算:

$$\begin{aligned} < H > &= (\frac{1}{\sqrt{2}})^2 E_1 + (\frac{1}{2})^2 E_2 + (-\frac{1}{2})^2 E_3 \\ &= 1 \end{aligned} \tag{42}$$

二是在Q表象下计算:

$$\begin{aligned} < H > &= <\psi|H|\psi > \\ &= \frac{1}{2} \begin{pmatrix} 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \\ &= 1 \end{aligned} \tag{43}$$

(3) 方法一，在H表象下直接求解:

$$\begin{aligned} |\psi(t) >= &= \frac{1}{\sqrt{2}} e^{-iE_1 t/\hbar}|\phi_1 > +\frac{1}{2} e^{-iE_2 t/\hbar}|\phi_2 > -\frac{1}{2} e^{-iE_3 t/\hbar}|\phi_3 > \\ &= \frac{1}{\sqrt{2}} e^{-i2t/\hbar}|\phi_1 > +\frac{1}{2} e^{-it/\hbar}|\phi_2 > -\frac{1}{2} e^{it/\hbar}|\phi_3 > \\ &= \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i2t/\hbar} \\ -isin(t/\hbar) \\ cos(t/\hbar) \end{pmatrix} \end{aligned} \tag{44}$$

方法二，计算$|\psi(t)>= e^{-iHt/\hbar}|\psi(0)>$, H 可写为

$$H = \begin{pmatrix} 2 & 0 \\ 0 & \sigma_x \end{pmatrix} \tag{45}$$

则

$$\begin{aligned} |\psi(t)> &= e^{-iHt/\hbar}|\psi> \\ &= \begin{pmatrix} e^{-i2t/\hbar} & 0 \\ 0 & e^{-i\sigma_x t/\hbar} \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \\ &= \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i2t/\hbar} & 0 \\ 0 & cos(t/\hbar) - i\sigma_x sin(t/\hbar) \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \\ &= \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i2t/\hbar} & 0 & 0 \\ 0 & cos(t/\hbar) & -isin(t/\hbar) \\ 0 & -isin(t/\hbar) & cos(t/\hbar) \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \\ &= \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i2t/\hbar} \\ -isin(t/\hbar) \\ cos(t/\hbar) \end{pmatrix} \end{aligned} \tag{46}$$

(4) Q表象到H表象的变换矩阵

$$S = \begin{pmatrix} <\phi_1|q_1> & <\phi_1|q_2> & <\phi_1|q_3> \\ <\phi_2|q_1> & <\phi_2|q_2> & <\phi_2|q_3> \\ <\phi_3|q_1> & <\phi_3|q_2> & <\phi_3|q_3> \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \tag{47}$$

3. 中微子振荡问题去年获得诺贝尔奖。假设中微子有两个态$|1>, |2>$,代表两种中微子。哈密顿量可以写为$H = h|1><1| + g|1><2| + g|2><1| + h|2><2|$。如果中微子初态是$|2>$，那么t时刻它是什么状态?

解：设t时刻状态为$|\psi(t)>$
方法一: H以$|1>, |2>$为基矢的矩阵表示为

$$\begin{aligned} H &= \begin{pmatrix} h & g \\ g & h \end{pmatrix} \\ &= hI_{2\times 2} + g\sigma_x \end{aligned} \tag{48}$$

将$e^{-\frac{i}{\hbar}Ht}$作用在初态上，得到:

$$\begin{aligned} |\psi(t)> &= e^{-iHt/\hbar}|2> \\ &= e^{-iht/\hbar}e^{-igt\sigma_x/\hbar}|2> \\ &= e^{-iht/\hbar}[cos(gt/\hbar) - i\sigma_x sin(gt/\hbar)]|2> \\ &= e^{-iht/\hbar}cos(gt/\hbar)|2> - e^{-iht/\hbar}isin(gt/\hbar)|1> \\ &= \begin{pmatrix} -e^{-iht/\hbar}isin\frac{gt}{\hbar} \\ e^{-iht/\hbar}cos\frac{gt}{\hbar} \end{pmatrix} \end{aligned} \tag{49}$$

方法二：将初态按H的本征矢展开. H的本征值和本征矢为

$$\begin{aligned} E_1 &= h+g, \quad |\phi_1>=\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ E_2 &= h-g, \quad |\phi_2>=\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{aligned} \tag{50}$$

初态可以写为H本征矢的叠加态，$|2>=a|\phi_1>+b|\phi_2>$, 其中

$$\begin{aligned} a &= <\phi_1|2>=\frac{1}{\sqrt{2}} \\ b &= <\phi_2|2>=-\frac{1}{\sqrt{2}} \end{aligned} \tag{51}$$

则

$$\begin{aligned} |\psi(t)> &= \frac{1}{\sqrt{2}}e^{-iE_1t/\hbar}|\phi_1>-\frac{1}{\sqrt{2}}e^{-iE_2t/\hbar}|\phi_2> \\ &= \frac{1}{\sqrt{2}}e^{-i(h+g)t/\hbar}|\phi_1>-\frac{1}{\sqrt{2}}e^{-i(h-g)t/\hbar}|\phi_2> \\ &= \begin{pmatrix} -e^{-iht/\hbar}isin\frac{gt}{\hbar} \\ e^{-iht/\hbar}cos\frac{gt}{\hbar} \end{pmatrix} \end{aligned} \tag{52}$$

4.设$|x\rangle, i=1,\ldots,N$,是A表象的一组正交归一完备基矢，现在选取另外的一个规范，基矢为$|x^{'}\rangle = |x\rangle\exp(i\phi_i), i=1,\ldots,N$.证明：(1)对任意态矢$|\alpha\rangle$,任意力学量Q的期望值不随规范不同而改变,但是$|\alpha\rangle$和Q的表示会不同.(2)Q的本征值不变，找到两个规范下本征矢的变换关系.

已知:$\sum\limits_i |i\rangle\langle i| = \boldsymbol{I}$,如果选取另外一组规范基矢$|i^{'}\rangle = |i\rangle\exp(i\phi_i), i=1,\ldots,N$,显然$\sum\limits_{i^{'}}|i^{'}\rangle\langle i^{'}| = \boldsymbol{I}$

(1) $\quad |\alpha\rangle = \sum\limits_i |i\rangle\langle i|\alpha\rangle = \sum\limits_{i^{'}}|i^{'}\rangle\langle i^{'}|\alpha\rangle$

力学量Q的期望值$\langle Q\rangle = \langle\alpha|\hat{Q}|\alpha\rangle$,从表达式就可以看出，Q的期望值与基的选取无关，可以在态矢与Q之间插入任意一组完备基矢，都不会影响Q的期望值。

在旧基矢中,$|\alpha\rangle = \begin{pmatrix} \langle 1|\alpha\rangle \\ \langle 2|\alpha\rangle \\ \vdots \\ \langle N|\alpha\rangle \end{pmatrix}$,在新基矢中,$|\alpha\rangle = \begin{pmatrix} \langle 1^{'}|\alpha\rangle \\ \langle 2^{'}|\alpha\rangle \\ \vdots \\ \langle N^{'}|\alpha\rangle \end{pmatrix} = \begin{pmatrix} \langle 1|\alpha\rangle e^{-i\phi_1} \\ \langle 2|\alpha\rangle e^{-i\phi_2} \\ \vdots \\ \langle N|\alpha\rangle e^{-i\phi_N} \end{pmatrix}$.

Q算符在旧基矢中的表示:

$$\begin{pmatrix} \langle 1|Q|1\rangle & \langle 1|Q|2\rangle & \cdots & \langle 1|Q|N\rangle \\ \langle 2|Q|1\rangle & \langle 2|Q|2\rangle & \cdots & \langle 2|Q|N\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle N|Q|1\rangle & \langle N|Q|2\rangle & \cdots & \langle N|Q|N\rangle \end{pmatrix}$$

Q算符在新基矢中的表示:

$$\begin{pmatrix} \langle 1^{'}|Q|1^{'}\rangle & \langle 1^{'}|Q|2^{'}\rangle & \cdots & \langle 1^{'}|Q|N^{'}\rangle \\ \langle 2^{'}|Q|1^{'}\rangle & \langle 2^{'}|Q|2^{'}\rangle & \cdots & \langle 2^{'}|Q|N^{'}\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle N^{'}|Q|1^{'}\rangle & \langle N^{'}|Q|2^{'}\rangle & \cdots & \langle N^{'}|Q|N^{'}\rangle \end{pmatrix}$$

显然：$\langle i^{'}|\hat{Q}|j^{'}\rangle = e^{-i(\phi_i-\phi_j)}\langle i|\hat{Q}|j\rangle$
变换矩阵为：

$$U = \begin{pmatrix} e^{i\phi_1} & & & \\ & e^{i\phi_2} & & \\ & & \ddots & \\ & & & e^{i\phi_N} \end{pmatrix}$$

任意态矢$|\alpha\rangle$与算符$\hat{Q}$的矩阵在新基矢中的表示与与旧基矢的关系为:

$$\begin{pmatrix} \langle 1^{'}|\alpha\rangle \\ \langle 2^{'}|\alpha\rangle \\ \vdots \\ \langle N^{'}|\alpha\rangle \end{pmatrix} = \begin{pmatrix} \langle 1|\alpha\rangle e^{-i\phi_1} \\ \langle 2|\alpha\rangle e^{-i\phi_2} \\ \vdots \\ \langle N|\alpha\rangle e^{-i\phi_N} \end{pmatrix} = \begin{pmatrix} e^{-i\phi_1} & & & \\ & e^{-i\phi_2} & & \\ & & \ddots & \\ & & & e^{-i\phi_N} \end{pmatrix} \begin{pmatrix} \langle 1|\alpha\rangle \\ \langle 2|\alpha\rangle \\ \vdots \\ \langle N|\alpha\rangle \end{pmatrix}$$

$$\begin{pmatrix} \langle 1^{'}|Q|1^{'}\rangle & \langle 1^{'}|Q|2^{'}\rangle & \cdots & \langle 1^{'}|Q|N^{'}\rangle \\ \langle 2^{'}|Q|1^{'}\rangle & \langle 2^{'}|Q|2^{'}\rangle & \cdots & \langle 2^{'}|Q|N^{'}\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle N^{'}|Q|1^{'}\rangle & \langle N^{'}|Q|2^{'}\rangle & \cdots & \langle N^{'}|Q|N^{'}\rangle \end{pmatrix} = \begin{pmatrix} e^{-i\phi_1} & & & \\ & e^{-i\phi_2} & & \\ & & \ddots & \\ & & & e^{-i\phi_N} \end{pmatrix} \begin{pmatrix} \langle 1|Q|1\rangle & \langle 1|Q|2\rangle & \cdots & \langle 1|Q|N\rangle \\ \langle 2|Q|1\rangle & \langle 2|Q|2\rangle & \cdots & \langle 2|Q|N\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle N|Q|1\rangle & \langle N|Q|2\rangle & \cdots & \langle N|Q|N\rangle \end{pmatrix}$$

$$\begin{pmatrix} e^{i\phi_1} & & & \\ & e^{i\phi_2} & & \\ & & \ddots & \\ & & & e^{i\phi_N} \end{pmatrix}$$

$Q^{'} = U^{\dagger}QU, U^{\dagger}U = \boldsymbol{I}$, 设$\boldsymbol{Q}\vec{q_i} = \lambda_i\vec{q_i}$,则$U^{\dagger}\boldsymbol{Q}UU^{\dagger}\vec{q_i} = \lambda_i U^{\dagger}\vec{q_i}$,即$\boldsymbol{Q'}\vec{q^{'}} = \lambda_i\vec{q^{'}}$,本征值不变，本征矢量在不同基下的变换关系为:$\vec{q^{'}} = U^{\dagger}\vec{q}$.