三维各向同性谐振子

中心力

$$V(r) = \frac{1}{2}\mu\omega^2 r^2$$

直角坐标： $V(x,y,z) = \frac{1}{2}\mu\omega^2(x^2+y^2+z^2)$

$$H = \frac{p_x^2}{2\mu} + \frac{1}{2}\mu\omega^2 x^2 + \frac{p_y^2}{2\mu} + \frac{1}{2}\mu\omega^2 y^2 + \frac{p_z^2}{2\mu} + \frac{1}{2}\mu\omega^2 z^2$$

$$= H_x + H_y + H_z$$

分离变量： 选 (H_x, H_y, H_z) 为守恒完全集

因为 $[H_x, H_y] = [H_x, H_z] = [H_y, H_z] = 0$.

$[H_x, H] = [H_y, H] = [H_z, H] = 0$.

$$\frac{dH_x}{dt} = \frac{dH_y}{dt} = \frac{dH_z}{dt} = 0.$$

设共本： $\psi_{n_x n_y n_z}(x,y,z) = \psi_{n_x}(x)\,\psi_{n_y}(y)\,\psi_{n_z}(z)$

$$H\psi_{n_x n_y n_z} = E\psi_{n_x n_y n_z}$$

$\Rightarrow$ $H_x\psi_{n_x} = E_{n_x}\psi_{n_x}$, $H_y\psi_{n_y} = E_{n_y}\psi_{n_y}$, $H_z\psi_{n_z} = E_{n_z}\psi_{n_z}$.

$E = E_{n_x} + E_{n_y} + E_{n_z}$

$E_{n_x} = (n_x + \frac{1}{2})\hbar\omega$, E_{n_y}, E_{n_z}.

$E = (N + \frac{3}{2})\hbar\omega$, $N = n_x + n_y + n_z = 0, 1, 2, \cdots$

给定 N，简并度 ~~$f = N \times (N-1)$~~

n_x 可取 $N+1$ 个值， $0, 1, 2, \cdots\cdots, N$

$n_y + n_z$ 可取 $N, N-1, \quad\quad , 0$

分配数： $N+1, N, \quad\quad , 1$

$$f = 1 + 2 + \cdots + (N+1) = \frac{1}{2}(N+1)(N+1+1)$$

以上 ω 各向同性。 也可以处理 $\omega_x \neq \omega_y \neq \omega_z$.

在球坐标系处理. $\psi(\vec{r}) = R(r)Y_{lm}(\theta,\varphi)$

$R(r) = \chi/r$

径向方程.

$$-\frac{\hbar^2}{2\mu}\chi'' + \left[V + \frac{\hbar^2 l(l+1)}{2\mu r^2}\right]\chi = E\chi$$

自然单位. $\hbar = \mu = \omega = 1$.

$$-\chi'' + r^2 + \frac{l(l+1)}{r^2} = 2E\chi \quad ①$$

给定 l.

边界条件: 当 $r\to 0$ 时. $\chi_l \sim r^{l+1}$

当 $r\to\infty$ 时. $\chi_l'' - r^2\chi_l = 0$.

$$\chi_l \sim e^{-\frac{r^2}{2}}$$

设 $\chi_l = r^{l+1}e^{-r^2/2}u(r)$.

代回①. $u'' + \frac{2}{r}(l+1-r^2)u' + [2E-(2l+3)]u = 0$.

令 $\xi^2 = r^2$, 得: $\xi\frac{d^2u}{d\xi^2} + \underbrace{\boxed{\left(l+\frac{3}{2}-\xi\right)}}_{\gamma-\xi}\frac{du}{d\xi} + \underbrace{\boxed{\left(\frac{E}{2} - \frac{l+\frac{3}{2}}{2}\right)}}_{-\alpha}u = 0$

$$\left(\frac{du}{2r\,dr} = \frac{du}{d\xi}\right.$$

$$\left.\xi\frac{d^2u}{d\xi^2} = -\frac{du}{4dr} + \frac{1}{4}\frac{d^2u}{dr^2}\right)$$

此为合流超几何方程.

$\alpha = \frac{1}{2}\left(l+\frac{3}{2}-E\right)$, $\gamma = l+\frac{3}{2}$.

解为: $F(\alpha,\gamma,\xi) = 1 + \frac{\alpha}{\gamma}\xi + \frac{\alpha(\alpha+1)}{\gamma(\gamma+1)}\frac{\xi^2}{2} + \cdots\frac{\xi^3}{3!}$

终止. $\alpha = -n_r$, $n_r = 0, 1, 2, \cdots$

给出. $E = \left(2n_r + l + \frac{3}{2}\right)\hbar\omega$,

令 $N = 2n_r + l$.

$E_N = \left(N+\frac{3}{2}\right)\hbar\omega$. $N = 0, 1, 2, \cdots$

完整径向波函数. $R_{n_r,l} = r^l e^{-\alpha^2 r^2/2} F(-n_r, l+\frac{3}{2}, \alpha^2 r^2)$.
（它里. $\alpha = \sqrt{\frac{\hbar}{\mu\omega}}$, 长度）

能级简并度. 给定 N.
l 的取: N, $N-2$, ……, 1（或 0）［由 N 奇偶］
n_r: 0, 1, 2, $\frac{N-1}{2}$（$\frac{N}{2}$）.

$N=$偶. $f = \sum_{l=0,2,\cdots,N} \underbrace{\boxed{2l+1}}_{m} = \frac{1}{2}(N+1)(N+2)$.

$N=$奇:

$\psi_{n_r l m}$ 给定 $n_r=0$, $\xrightarrow{l=1.} \psi_{011}, \psi_{01-1}, \psi_{010}$

$\phi_{n_x n_y n_z}$, $\xrightarrow{N=1.} \phi_{100}, \phi_{010}, \phi_{001}$.