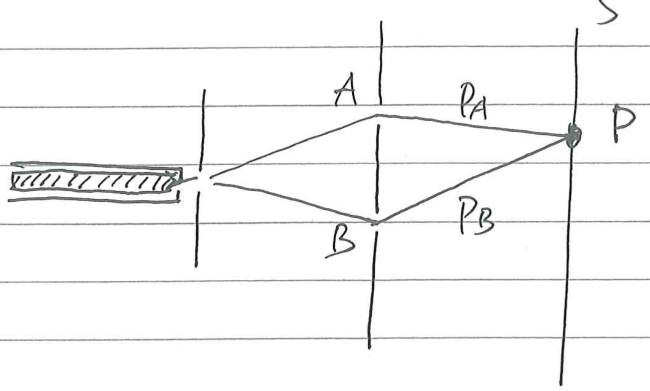
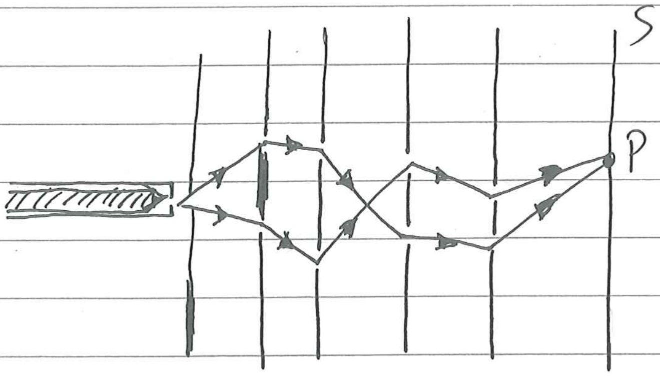
路径积分量子化



电子枪发射一个电子. 可以通过路径 P_A 到~~达~~达 P点
也可以通过路径 P_B 到达 P点.

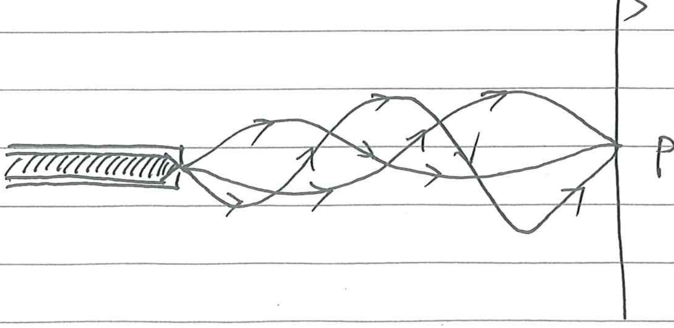
几率幅分别为 a_A 与 a_B. 设 $a_A/a_B \propto e^{i\varphi}$.

在P点几率 $|a_A+a_B|^2 = |a_A|^2+|a_B|^2+2|a_A||a_B|\cos\varphi$ → 干涉条纹



推广: 很多m个遮挡屏, 每个屏上n条狭缝. 一条路径 a_c, c标记路径. 由屏与缝的组合确定.
P点找到电子的几率:
$$\left|\sum_c a_c\right|^2$$

假设: $m\to\infty$, $n\to\infty$. 屏就没了!



对任意一条 $\boxed{\text{路径积分}}$ 给出到达P点的几率!

我们暂时考虑一个粒子的一维运动，比如谐振子

$$i\hbar \frac{\partial |\psi\rangle}{\partial t} = H|\psi\rangle.$$

波函数：$$i\hbar \frac{\partial \langle x|\psi\rangle}{\partial t} = \langle x|H|\psi\rangle$$

$$i\hbar \frac{\partial \psi(x,t)}{\partial t} = H\left(x, -i\hbar\frac{\partial}{\partial x}\right)\psi(x,t)$$

引入时间演化算符．$|\psi(t')\rangle = U(t',t)|\psi(t)\rangle$

$$U(t',t) = e^{-\frac{i\hat{H}(t'-t)}{\hbar}}$$

$$\psi(x',t') = \langle x'|\psi(t')\rangle = \int \langle x'|U(t',t)|x\rangle\langle x|\psi(t)\rangle\,dx$$

$$= \int U(x',t';x,t)\,\psi(x,t)\,dx.$$

把 $t'-t$ 分成 N 分：$t'-t = N\Delta t$，记 $t_k = t + k\Delta t$，$k=0,\cdots N$

$$U = e^{-\frac{iH(t-t')}{\hbar}} = e^{-\frac{iH}{\hbar}(t'-t_{N-1})} \cdot e^{-\frac{iH}{\hbar}(t_{N-1}-t_{N-2})} \cdot \cdots e^{\cdots}$$

$$\cdots e^{-\frac{iH}{\hbar}(t_1-t)}$$

$$= U(t',t_{N-1})\cdots U(t_1-t)$$

$$U(x',t';x,t) = \int dx_{N-1}\,dx_{N-2}\cdots dx_1\,U(x',t';x_{N-1},t_{N-1})$$

$$\cdot U(x_{N-1},t_{N-1};x_{N-2},t_{N-2})\cdots U(x_1,t_1;x,t)$$

其中 $$U(x_{k+1},t_{k+1};x_k,t_k) = \langle x_{k+1}|e^{-\frac{i}{\hbar}H\Delta t}|x_k\rangle$$

$$\approx \langle x_{k+1}|\left(1-\frac{i}{\hbar}H\Delta t\right)|x_k\rangle$$

这里 $$\langle x_{k+1}|H|x_k\rangle = \int dP_k \underbrace{\langle x_{k+1}|P_k\rangle\langle P_k|}_{\text{完备性}}H|x_k\rangle$$

由于 H是 $\hat{p}, \hat{x}$ 的函数. 所以

$$\langle p_k | H(\hat{p}, \hat{x}) | x_k \rangle = H(p_k, x_k) \langle p_k | x_k \rangle$$

$$\langle p_k | x_k \rangle = \frac{e^{-i\frac{p_k x_k}{\hbar}}}{\sqrt{2\pi\hbar}}$$

$$U(x_{k+1}, t_{k+1}; x_k, t_k) = \int dp_k \langle x_{k+1} | p_k \rangle \langle p_k | x_k \rangle \left(1 - \frac{i}{\hbar} H(p_k, x_k) \Delta t\right)$$

$$= \int \frac{dp_k}{2\pi\hbar} e^{i\frac{p_k}{\hbar}(x_{k+1} - x_k) - \frac{i\Delta t}{\hbar} H(p_k, x_k)}$$

那么

$$U(x', t'; x, t) = \int \frac{dp_{N-1} \cdots dp_0}{(2\pi\hbar)^N} \int dx_{N-1} \cdots dx_1 \times e^{\frac{i}{\hbar} \sum_{k=0}^{N-1} \left[p_k (x_{k+1} - x_k) - \Delta t H(p_k, x_k)\right]}$$

取极限: $N \to \infty$, $\Delta t \to 0$, 那么 $x_{k+1} - x_k = \dot{x}(t) \Delta t$

$$\sum_{k=0}^{N-1} \Delta t = \int_t^{t'} dt$$

公式(A) $\therefore U(x', t'; x, t) = \int_{\substack{x(t) = x \\ x(t') = x'}} \mathcal{D}p(t) \mathcal{D}x(\tau) \; e^{\frac{i}{\hbar} \underbrace{\boxed{\int_t^{t'} d\tau \left[p(\tau)\dot{x}(\tau) - H(p(\tau), x(\tau))\right]}}_{\text{“作用量 } S\text{”}}}$

回到经典力学!

$$\begin{cases} \dfrac{dx}{dt} = \dfrac{\partial H(p, x)}{\partial p} \\[2ex] \dfrac{dp}{dt} = \dfrac{-\partial H}{\partial x} \end{cases}$$

~~拉普拉斯量~~. 拉格朗日量. $\boxed{L(x, \dot{x}; p) = p\dot{x} - H(p, x)}$

作用量 (action) $S = \int dt \, L(x, \dot{x}; p)$

S是 L 的泛函(functional): $f: x \in R \to f(x) \in R$

$F: f(x) \in H \to F(\{f(x)\}) \in R$

理解: $F(\{f(x)\}) \to F(f(x_1), \cdots\cdots, f(x_N))$ 无穷多元函数.

最小作用量原理: $\delta S=0$ 给出运动方程.

$$\delta S=\int dt\left\{\delta x(t)\cdot\frac{\partial L}{\partial x}+\delta\dot{x}\cdot\frac{\partial L}{\partial \dot{x}}+\delta p\cdot\frac{\partial L}{\partial p}\right\}$$

$$=\int dt\left\{\delta x(t)\cdot\underbrace{\left[\frac{\partial L}{\partial x}-\frac{d}{dt}\frac{\partial L}{\partial \dot{x}}\right]}_{\delta S/\delta x}+\delta p\cdot\underbrace{\frac{\partial L}{\partial p}}_{\delta S/\delta p}\right\}$$

运动方程由任意变化 δx, δp, 有 $\delta S=0$. 给出:

① $\frac{\partial L}{\partial x}=\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} \Rightarrow \frac{dp}{dt}=-\frac{\partial H}{\partial x}$

② $\frac{\partial L}{\partial p}=0 \Rightarrow \frac{dx}{dt}=\frac{\partial H}{\partial p}$

或者. $H(x,p)=\frac{p\cdot\dot{x}}{2}+V(x) \Rightarrow L=+\frac{p\cdot\dot{x}}{2}-V(x)$.

① 就是拉格朗日方程.

回到量子力学: $\boxed{S'=\int_t^{t'}d\tau\left[p(\tau)\dot{x}(\tau)-H(p(\tau),x(\tau))\right]}$

与经典力学一致.

但是每一条路径(path) 有一个"振幅" $=e^{\frac{i}{\hbar}S'}$

$x(\tau)$, $p(\tau)$ 任意, 独立, 不保证 $p(\tau)=m\dot{x}(\tau)$.

由于 S 中对 $p(\tau)$ 的依赖 取决于

$$\int_t^{t'}\left[p(\tau)\dot{x}(\tau)-\frac{p(\tau)^2}{2m}\right]d\tau=\int_t^{t'}\left(-\frac{1}{2m}(p(\tau)-m\dot{x}(\tau))^2+\frac{m\dot{x}(\tau)^2}{2}\right)d\tau$$

引入对 $U(x',t';x,t)$ 中的 $\mathcal{D}p$ 积分. (高斯积分)

$$\int\prod_\tau dp(\tau)=\prod_\tau\int dp(\tau)=\prod\sqrt{\frac{m}{2\pi i\hbar\Delta t}}$$

(公式B) $U(x',t';x,t)=\int_{x(t)=x}^{x(t')=x'}\mathcal{D}x(\tau)\,e^{\frac{i}{\hbar}S}$, $\boxed{S'=\int_t^{t'}d\tau\left(\frac{m\dot{x}(\tau)^2}{2}-V(x(\tau))\right)}$