最小作用量原理：　　$\delta S=0$　给出运动方程.

$$\delta S=\int dt\left\{\delta x(t)\cdot\frac{\partial L}{\partial x}+\delta\dot{x}\cdot\frac{\partial L}{\partial\dot{x}}+\delta p\cdot\frac{\partial L}{\partial p}\right\}$$

$$=\int dt\left\{\delta x(t)\cdot\underbrace{\left[\frac{\partial L}{\partial x}-\frac{d}{dt}\frac{\partial L}{\partial\dot{x}}\right]}_{\delta S/\delta x}+\delta p\cdot\underbrace{\frac{\partial L}{\partial p}}_{\delta S/\delta p}\right\}$$

运动方程由任意变化 δx, δp，有 $\delta S=0$. 给出：

① $\frac{\partial L}{\partial x}=\frac{d}{dt}\frac{\partial L}{\partial\dot{x}}\Rightarrow\frac{dp}{dt}=-\frac{\partial H}{\partial x}$

② $\frac{\partial L}{\partial p}=0\Rightarrow\frac{dx}{dt}=\frac{\partial H}{\partial p}$

或者. $H(x,p)=\frac{p\cdot\dot{x}}{2}+V(x)\Rightarrow L=+\frac{p\cdot\dot{x}}{2}-V(x)$.

① 就是拉格朗日方程.

回到量子力学：

$$\boxed{S'=\int_t^{t'}d\tau\left[p(\tau)\dot{x}(\tau)-H(p(\tau),x(\tau))\right]}$$

与经典力学一致.

但是每一条路径(path) 有一个"振幅" $=e^{\frac{i}{\hbar}S'}$

$x(\tau)$, $p(\tau)$ 任意，独立，不保证 $p(\tau)=m\dot{x}(\tau)$.

由于 S 中对 $p(\tau)$ 的依赖取决于

$$\int_t^{t'}\left[p(\tau)\dot{x}(\tau)-\frac{p(\tau)^2}{2m}\right]d\tau=\int_t^{t'}\left(-\frac{1}{2m}(p(\tau)-m\dot{x}(\tau))^2+\frac{m\dot{x}(\tau)^2}{2}\right)d\tau$$

进行对 $U(x',t';x,t)$ 中的 $\mathcal{D}p$ 积分.（高斯积分）.

$$\int\prod_\tau dp(\tau)=\prod_\tau\int dp(\tau)=\prod\sqrt{\frac{m}{2\pi i\hbar\Delta t}}$$

(公式B) $U(x',t';x,t)=\int_{x(t)=x}^{x(t')=x'}\mathcal{D}x(\tau)\,e^{\frac{i}{\hbar}S}$,　$\boxed{S'=\int_t^{t'}d\tau\left(\frac{m\dot{x}(\tau)^2}{2}-V(x(\tau))\right)}$

公式A与B都对，但A更基本．因为这可解释为Berry phase
也有情况无法积掉 $p(\tau)$　　（以后讲）

$\delta S=0 \Rightarrow$ 经典方程．最小作用量．

- 量子力学：每条路径都有贡献．但是如果ℏ很小，S变一点 phase $\frac{iS}{\hbar}$ 变很大！平均起来为0．除了 $\delta S=0$ 的路径．

- 无算符．代价：无穷维积分．
- 推导 $[\hat{x}, \hat{p}]=i\hbar$．

设一个物理量 $F(x,p)$．计算它的"路径平均"

$$\langle F(\{x\},\{p\})\rangle \equiv \int \mathcal{D}x(\tau)\,\mathcal{D}p(\tau)\, F(\{x\},\{p\})\, e^{\frac{i}{\hbar}S(\{x\},\{p\})}$$

改写 $F(\{x\},\{p\}) \longrightarrow F(x_1,\cdots,x_{N-1};\, p_0, p_1,\cdots,p_{N-1}) \equiv F(x_k, p_k)$

$$\int \mathcal{D}x\,\mathcal{D}p \to \prod_k \int dx_k\, dp_k$$

变换：$x_k=\tilde{x}_k+y_k$，$p_k=\tilde{p}_k+\zeta_k$；y_k, ζ_k 无穷小

$$\langle F(x_k,p_k)\rangle = \int \prod_k d\tilde{x}_k\, d\tilde{p}_k\, F(\tilde{x}_k+y_k, \tilde{p}_k+\zeta_k) \times e^{\frac{i}{\hbar}S(\tilde{x}_k+y_k,\, \tilde{p}_k+\zeta_k)}$$

$$= \int \prod_k d\tilde{x}_k\, d\tilde{p}_k \left(F+\sum_k y_k \frac{\partial F}{\partial x_k}+\sum_k \zeta_k \frac{\partial F}{\partial p_k}\right) \times \left(1+\frac{i}{\hbar}\sum_k y_k \frac{\partial S}{\partial x_k}+\frac{i}{\hbar}\sum_k \zeta_k \frac{\partial S}{\partial p_k}\right) e^{iS/\hbar}$$

$$= \langle F(x_k,p_k)\rangle + \sum_k y_k \left\langle \frac{\partial F}{\partial x_k}+\frac{i}{\hbar}F\frac{\partial S}{\partial x_k}\right\rangle + \sum_k \zeta_k \left\langle \frac{\partial F}{\partial p_k}+\frac{i}{\hbar}F\frac{\partial S}{\partial p_k}\right\rangle$$

由于 η_k 与 ξ_k 任意.

$$\left\langle \frac{\partial F}{\partial x_k} \right\rangle = -\frac{i}{\hbar}\left\langle F\frac{\partial S}{\partial x_k} \right\rangle \quad ①$$

$$\left\langle \frac{\partial F}{\partial p_k} \right\rangle = -\frac{i}{\hbar}\left\langle F\frac{\partial S}{\partial p_k} \right\rangle \quad ②$$

如果: $F(x_k, p_k)$ 是 $p(k_0)$ (只取对 k_0 处 p).

由② $1 = -\frac{i}{\hbar}\left\langle p(k_0)\frac{\partial S}{\partial p_{k_0}} \right\rangle$

由于 $S = \sum_k p_k(x_{k+1} - x_k) - \Delta t \sum_k H(p_k, x_k)$

有: $\frac{\partial S}{\partial p_{k_0}} = x_{k_0+1} - x_{k_0} - \Delta t \frac{\partial H}{\partial p_{k_0}}$

当 $\Delta t \to 0$, $\boxed{i\hbar = \langle p(k_0)(x_{k_0+1} - x_{k_0}) \rangle}$ (C)

前面推导

$\langle x'_{k+1} | x\, p | x_k \rangle$
$= \langle x'_{k+1} | x | p_k \rangle \langle p_k | p | x_k \rangle$

$\langle x' | p\, x | x_k \rangle = \langle x' | p | p_k \rangle \langle p_k | x | x_k \rangle$

$\langle x p \rangle_{t=t_0} = \langle x_{k_0+1} p_{k_0} \rangle$

$\langle p x \rangle_{t=t_0} = \langle p_{k_0} x_{k_0} \rangle$

因此. (C) 式是等时对易式: $[\hat{x}, \hat{p}] = i\hbar$.