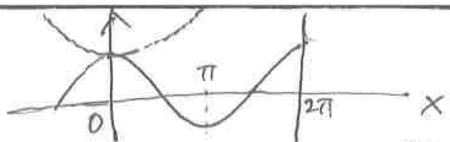


余弦 → 双曲余弦

理解：$\cos(x) = \dfrac{e^{ix}+e^{-ix}}{2} = \dfrac{e^{\tau}+e^{-\tau}}{2} = ch(\tau)$

x实，τ虚；但是，τ是实的话，也是个函数！

虚时情况：

$$|\psi(t')\rangle = U(t',t)|\psi(t)\rangle = e^{-\frac{i}{\hbar}H\cdot(t'-t)}|\psi(t)\rangle$$

若 $\left.\begin{array}{l} it = \tau \\ t = -i\tau \end{array}\right\}$ 即 τ 是虚时间. $\boxed{e^{-\frac{i}{\hbar}H\cdot t} = e^{-\frac{H}{\hbar}\tau}}$ τ = 时间×i

$\boxed{t\text{ 是实 则 }\tau\text{ 是虚}} \to \boxed{\tau\text{ 是实，}t\text{ 就是虚的}}$

重新看 $\langle x'|U(t',t)|x\rangle$ 用 τ 表示（$it'=\tau'$，$t=0$）

$$\langle x'|e^{-\frac{H\tau'}{\hbar}}|x\rangle = \int_{x(0)=x}^{x(\tau')=x'} \mathcal{D}x(\tau)\int\mathcal{D}p(\tau)\, e^{-\frac{1}{\hbar}S} \leftarrow e^{\frac{i}{\hbar}S(t,t')}$$

$$\boxed{-\frac{1}{\hbar}S = \frac{1}{\hbar}\int_0^{\tau'} -ip(\tau)\dot{x}(\tau) + H(p(\tau),x(\tau))\,d\tau} \longrightarrow (A)$$

（对比 $\frac{i}{\hbar}S = \frac{i}{\hbar}\int_t^{t'}[p(t'')\dot{x}(t'') - H(p(t''),x(t''))]\,dt''$. 实时）.

由于 $\boxed{\int_t^{t'}dt'' \to \int_0^{\tau'}(-i\,d\tau)}$；$\boxed{\dot{x}(t'') = \frac{dx(t'')}{dt''} \longrightarrow \frac{dx(\tau)}{-i\,d\tau} = i\dot{x}(\tau)}$

积掉 p 的情况 $\to i\int_t^{t'}dt'' \to \int_0^{\tau'}d\tau$ $\boxed{\frac{i}{\hbar}S = \frac{i}{\hbar}\int_t^{t'}\left[\frac{m\dot{x}(t'')^2}{2} - V(x(t''))\right]dt''}$

设 $\frac{\tau'}{\hbar} = \beta$. 并考虑特殊情况 $x'=x$，对 x 积分.

我们有：$\mathrm{Tr}\, e^{-\beta H} = \int dx \boxed{\langle x|e^{-\beta H}|x\rangle} \longrightarrow Z$

$$= \int dx \int_{x(0)=x}^{x(\tau')=x}\mathcal{D}x(\tau)\int\mathcal{D}p(\tau)\, e^{-\frac{1}{\hbar}S} \quad \text{（将 } p(\tau) \text{ 积掉）}$$

$$= \int dx \int_{x(0)}^{x(\tau')}\mathcal{D}x(\tau)\, e^{-\frac{1}{\hbar}\int_0^{\tau'}\left[\frac{m\dot{x}(\tau)^2}{2}+V(x(\tau))\right]d\tau}$$

$\boxed{\begin{array}{c}\frac{d\tau}{\hbar} \to d\tilde{\tau} \\ x(\beta) = x(0)\end{array}}$

$$= \boxed{\int dx \int_{x(0)}^{x(\beta)}\mathcal{D}x(\tilde{\tau})\, e^{-\int_0^{\beta}\left[\frac{m\dot{x}(\tilde{\tau})^2}{2}+V(x(\tilde{\tau}))\right]d\tilde{\tau}}}$$

(B)

统计物理！

注意：(A)式中 i 并没有消失，虽然我们已经用 $\tau=it$
这一项给出一个"相位"(phase)
然而，积掉 $p(\tau)$ 后，相位也消失了，此时 (B) 对应
- "经典"统计物理的配分函数 ⟶ 量子统计的"经典"情况.
- $x \rightarrow$ 弦 $x(\tau)$ ~~d维~~ ⟹ d维量子系统 $\xrightarrow{\text{对应}}$ d+1 维经典系统.
- 如果那个相位还在，不能对应 d+1 维经典系统.

利用虚时演化路径积分求解实时问题

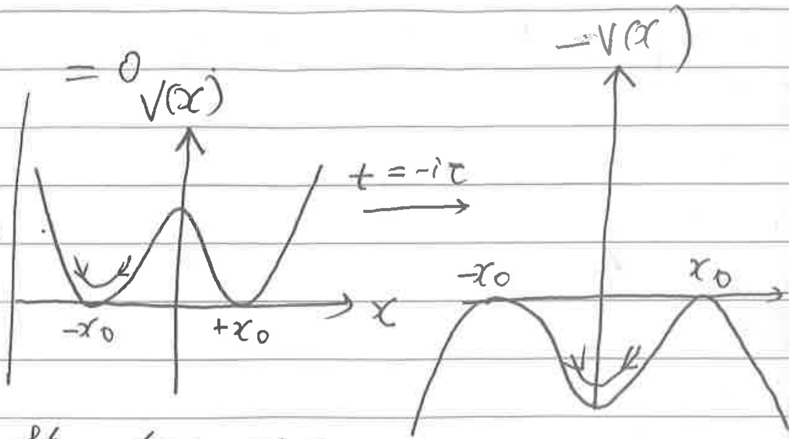
现在我们问：对 (B) 贡献最大的路径是哪条？

$$\delta\int_0^\beta\left[\frac{m\dot{x}(\tilde{\tau})^2}{2}+V(x(\tilde{\tau}))\right]d\tilde{\tau}=0 .$$

$$-\frac{d}{d\tilde{\tau}}\frac{\partial\left(\frac{m\dot{x}^2}{2}\right)}{\partial\dot{x}}+\frac{\partial V}{\partial x}=0$$

$$-m\ddot{x}+\frac{\partial V}{\partial x}=0$$

$$m\ddot{x}=\frac{\partial V}{\partial x}$$

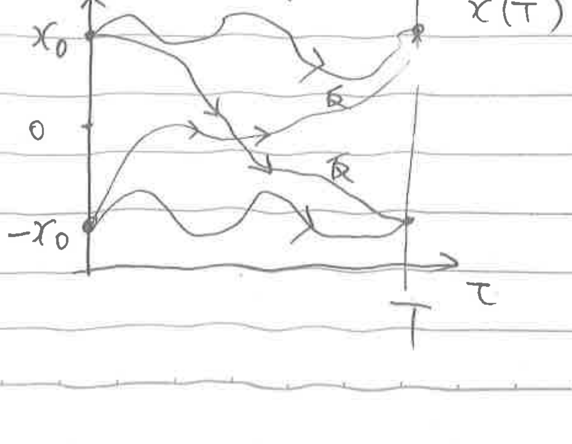


经典运动方程，但是势的符号相反.
解称为瞬子 (instanton).

设 $\hbar=1$, $\tau=T$ large

从 $x_0\to x_0$　$\langle x_0|e^{-HT}|x_0\rangle=\langle -x_0|e^{-HT}|x_0\rangle$

从 $x_0\to -x_0$, $\langle x_0|e^{-HT}|-x_0\rangle=\langle -x_0|e^{-HT}|x_0\rangle$



instanton and anti-instanton

自旋系统的路径积分.

自旋 1/2

$$|\psi\rangle = \alpha|\uparrow\rangle + \beta|\downarrow\rangle = \begin{pmatrix}\alpha\\ \beta\end{pmatrix}, \quad |\alpha|^2+|\beta|^2=1.$$

自由度 其实有3个.

$$|\psi\rangle = |b,\theta,\varphi\rangle = e^{ib}\left(e^{-\frac{i\varphi}{2}}\cos\frac{\theta}{2}|\uparrow\rangle + e^{\frac{i\varphi}{2}}\sin\frac{\theta}{2}|\downarrow\rangle\right)$$

e^{ib} 为整体相因子. 无关乎物理. gauge 变换的自由度

$(\cdot)$ 中是 $\sigma_{\vec{n}}$ 的本征态. (本征值为 1). 完备:

$$\int_0^{\pi}\sin\theta\, d\theta\int_0^{2\pi}d\varphi\ |b,\theta,\varphi\rangle\langle b,\theta,\varphi| = |\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow| = \mathbb{1}$$

$$\langle b,\theta,\varphi|\frac{\hbar}{2}\vec{\sigma}|b,\theta,\varphi\rangle = \frac{1}{2}\langle\vec{n}|\vec{\sigma}|\vec{n}\rangle = \frac{1}{2}\vec{n} \qquad (\hbar=1)$$

但是不正交. $\langle\vec{n}'|\vec{n}\rangle \neq \delta(\vec{n}-\vec{n}')$

对路径积分而言, 完备性有了就 OK.

令 $|\tau\rangle = |b(\tau),\theta(\tau),\varphi(\tau)\rangle$

$$\langle\tau+\Delta\tau|e^{-\Delta\tau H}|\tau\rangle \cong \langle\tau+\Delta\tau|(1-\Delta\tau H)|\tau\rangle$$

$$= \langle\tau+\Delta\tau|\tau\rangle - \Delta\tau\langle\tau+\Delta\tau|H|\tau\rangle$$

$$\boxed{\text{由于 } 0 = \frac{d}{d\tau}\langle\tau|\tau\rangle = \langle\dot{\tau}|\tau\rangle + \langle\tau|\dot{\tau}\rangle = 2\,\mathrm{Re}\langle\tau|\dot{\tau}\rangle}$$

$$\approx \left[\langle\tau| + \Delta\tau\frac{d}{d\tau}\langle\tau|\right]|\tau\rangle - \Delta\tau\langle\tau|H|\tau\rangle \qquad (\text{略去 } \Delta\tau^2)$$

$$= 1 + \Delta\tau\left[\langle\dot{\tau}|\tau\rangle - \langle\tau|H|\tau\rangle\right]$$

$$= e^{\Delta\tau\left[\langle\dot{\tau}|\tau\rangle - \langle\tau|H|\tau\rangle\right]}$$

$\langle\dot{\tau}|\tau\rangle$ 纯虚数 $\longrightarrow$ 相位.

反映 $|\tau\rangle$ 与 $|\tau+\Delta\tau\rangle$ 的"连络"