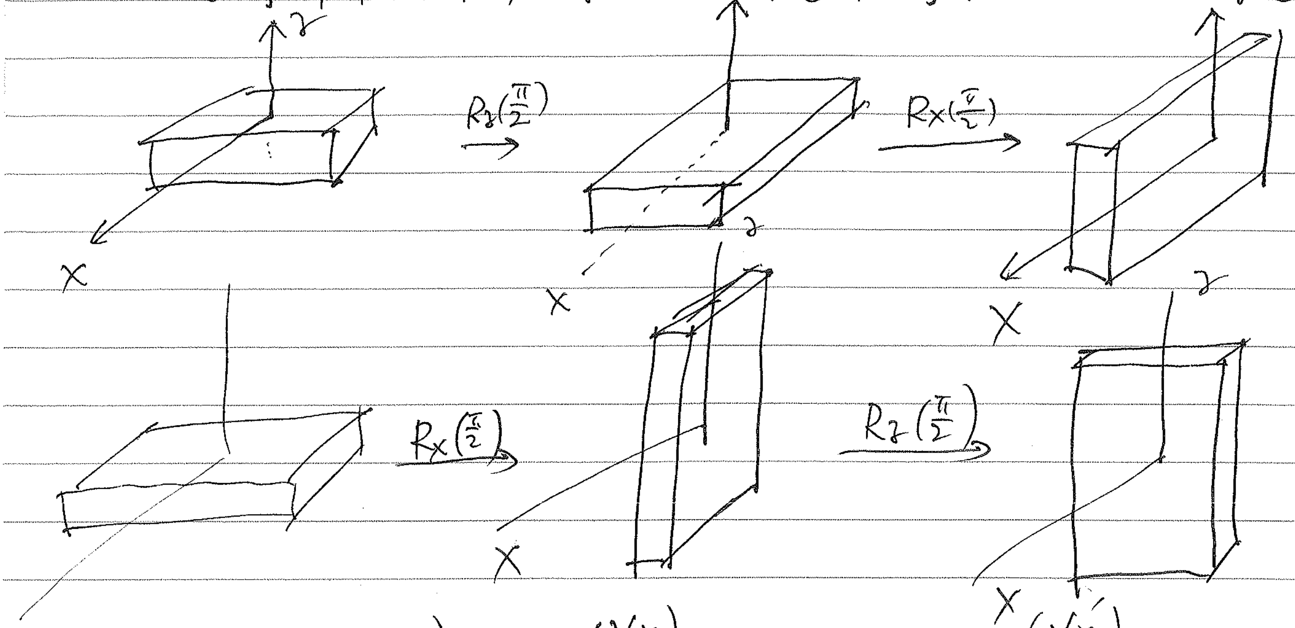
角动量理论

1. 旋转与角动量对易关系

绕同一个轴的转动对易。 绕不同轴的转动不对易。



一个矢量 $\vec{V} = \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix}$ 转动为 $\begin{pmatrix} V_x' \\ V_y' \\ V_z' \end{pmatrix} = R \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix}$

$RR^T = R^T R = 1$, 正交矩阵, 保证 $|\vec{V}| = |\vec{V}'|$

证明: $|V|^2 = \vec{V}^T V = \vec{V}'^T R^T R \vec{V}'$, 当 $R^T R = 1$ 时, $|\vec{V}| = |\vec{V}'|$

考虑 $R_z(\phi)$ 绕z轴逆时针转 ϕ.

$$R_z(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(坐标不动)

无穷小

$$R_z(\varepsilon) = \begin{pmatrix} 1-\frac{\varepsilon^2}{2} & -\varepsilon & 0 \\ \varepsilon & 1-\frac{\varepsilon^2}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

忽略 ε^3 以上 $O(\varepsilon^3)$

另外一种约定: 把坐标轴顺时针转 ϕ. 与上面等价.

此为"被动"旋转　　　　主动旋转

类似：
$$R_x(\varepsilon)=\begin{pmatrix}1 & 0 & 0\\ 0 & 1-\frac{\varepsilon^2}{2} & -\varepsilon\\ 0 & \varepsilon & 1-\frac{\varepsilon^2}{2}\end{pmatrix}$$

可以由 $x\to y\to z\to x$ 的轮转关系得到

$$R_y(\varepsilon)=\begin{pmatrix}1-\frac{\varepsilon^2}{2} & 0 & \varepsilon\\ 0 & 1 & 0\\ -\varepsilon & 0 & 1-\frac{\varepsilon^2}{2}\end{pmatrix}$$

考虑：绕y转后再绕x转

$$R_x(\varepsilon)R_y(\varepsilon)=\begin{pmatrix}1-\frac{\varepsilon^2}{2} & 0 & \varepsilon\\ \varepsilon^2 & 1-\frac{\varepsilon^2}{2} & -\varepsilon\\ -\varepsilon & \varepsilon & 1-\varepsilon^2\end{pmatrix}$$

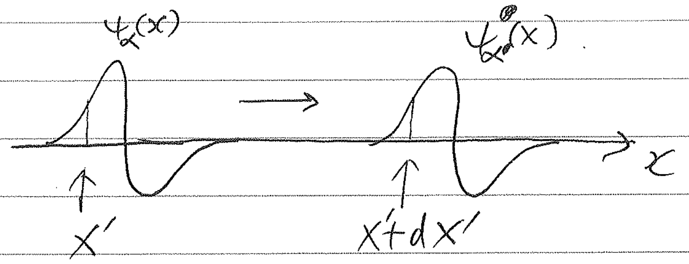
与 绕x转后再绕y转.

$$R_y(\varepsilon)R_x(\varepsilon)=\begin{pmatrix}1-\frac{\varepsilon^2}{2} & \varepsilon^2 & \varepsilon\\ 0 & 1-\frac{\varepsilon^2}{2} & -\varepsilon\\ -\varepsilon & \varepsilon & 1-\varepsilon^2\end{pmatrix}$$

$$[R_x(\varepsilon), R_y(\varepsilon)]=\begin{pmatrix}0 & -\varepsilon^2 & 0\\ \varepsilon^2 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}=R_z(\varepsilon^2)-\underset{\downarrow\ R_{any}(0)}{1}$$

量子力学中的无穷小转动

• 先考虑空间平移： 把波函数向右移动 dx.



$$\psi_\alpha(x') = \langle x'|\alpha\rangle$$
$$= \langle x'+dx'|\alpha'\rangle = \psi_{\alpha'}(x'+dx')$$

$$|\alpha\rangle \xrightarrow{T(dx')} |\alpha'\rangle = T(dx')|\alpha\rangle = T(dx')\int dx'\,|x'\rangle\langle x'|\alpha\rangle$$

$$= \int dx'\,|x'+dx'\rangle\langle x'|\alpha\rangle$$

以 $x'+dx'$ 为 x''，那么，$x' = x''-dx'$

$$|\alpha'\rangle = \int dx'\,|x'\rangle\langle x'-dx'|\alpha\rangle$$

也就是 $\psi_\alpha(x'-dx') = \psi_{\alpha'}(x')$

$$\boxed{T(dx')|x'\rangle = |x'+dx'\rangle}$$

1. 由于 $\langle\alpha|\alpha\rangle=1$，平移后也要 $\langle\alpha'|\alpha'\rangle=1$.

而 $\langle\alpha'|\alpha'\rangle = \langle\alpha|T^\dagger\, T|\alpha\rangle$

$\therefore\ T^\dagger T = 1$ ⟶ 平移算符是幺正的.

2. 两次相继操作.

$$T(dx'')\,T(dx') = T(dx'+dx'') \rightarrow \text{乘法}$$

3. 逆操作.

$$T(-dx') = T^{-1}(dx') \rightarrow \text{反向操作＝逆}$$

(回复到 $T(dx')$ 之前

4. 单位操作 $\lim\limits_{dx'\to 0} T(dx') = 1$

以上是"群" group.

假设无穷小平移可以写为

$$T(d\vec{x}') = 1 - i\vec{K}\cdot d\vec{x}'$$

厄密算符 $\vec{K}=(K_x, K_y, K_z)$

那么以上4点都可以满足.

① $T^{\dagger} T = (1+i\vec{K}^{\dagger}\cdot d\vec{x}')(1-i\vec{K}\cdot d\vec{x}') = 1-i(\vec{K}-\vec{K}^{\dagger})\cdot d\vec{x}' + O(dx'^2)$

由于 $\vec{K}^{\dagger}=\vec{K}$. $\therefore T^{\dagger}T \approx 1$.

②. $T(d\vec{x}'')T(d\vec{x}') = (1-i\vec{K}\cdot d\vec{x}'')(1-i\vec{K}\cdot d\vec{x}')$

$$= 1-i\vec{K}\cdot(d\vec{x}''+d\vec{x}') = T(d\vec{x}'+d\vec{x}'')$$

③. ④ 显然

接受此假设. 我们来推导基本对易式

$$\hat{x}\,T(dx')|x'\rangle = \hat{x}|x'+dx'\rangle = (x'+dx')|x'+dx'\rangle$$

$$T(dx')\,\hat{x}\,|x'\rangle = x'T(dx')|x'\rangle = x'|x'+dx'\rangle$$

因此: $[\hat{x}, T(dx')]\,|x'\rangle = dx'|x'+dx'\rangle \approx dx'|x'\rangle$

$|x'\rangle$ 任意 $\therefore$ $[\hat{x}, T(dx')] = dx'$

代入 $T(d\vec{x}') = 1-i\vec{K}\cdot d\vec{x}'$

$$-i\hat{x}\,\vec{K}\cdot d\vec{x}' + i\vec{K}\cdot d\vec{x}'\,\hat{x} = d\vec{x}'$$

$$[\hat{x}_i, \hat{K}_j] = i\delta_{ij}$$

考虑量纲, $\hat{K} = \hat{P}/\hbar$ (波矢)

$\hat{P}$ 就是量子力学中的动量算符 $\boxed{[x_i, p_j]=i\hbar\delta_{ij}}$

对于有限的平移 $\Delta x'$ 等于累积无穷小平移 dx'.

$$T(\Delta x')|x'\rangle = |x'+\Delta x'\rangle$$

$$= T\left(\frac{\Delta x'}{N}+\cdots\cdots+\frac{\Delta x'}{N}\right)|x'\rangle$$

$$= \lim_{N\to\infty}\left(1-\frac{iP_x}{\hbar}\cdot\frac{\Delta x'}{N}\right)^N = e^{-\frac{iP_x}{\hbar}\Delta x'}$$

$\hat{p}$ 是平移操作的"生成元" $\boxed{\text{generator}}$.

$$\left(1-\frac{i\hat{P}\Delta x'}{\hbar}\right)|\alpha\rangle = \int dx'\, T(\Delta x')\,|x'\rangle\langle x'|\alpha\rangle$$

$$= \int dx'\,|x'\rangle\langle x'-\Delta x'|\alpha\rangle$$

$$= \int dx'\,|x'\rangle\left(\langle x'|-\Delta x'\frac{\partial}{\partial x'}\langle x'|\right)|\alpha\rangle$$

$$= \int dx'\,|x'\rangle\left(\langle x'|\alpha\rangle-\Delta x'\frac{\partial}{\partial x'}\langle x'|\alpha\rangle\right)$$

以 $\langle x''|\cdot\left(1-\frac{i\hat{P}\Delta x'}{\hbar}\right)|\alpha\rangle$.

$$\langle x''|\left(-\frac{i\hat{P}}{\hbar}\Delta x'\right)|\alpha\rangle = -\Delta x'\frac{\partial}{\partial x''}\langle x''|\alpha\rangle$$

$$\left(-i\frac{\Delta x'}{\hbar}\right)\hat{P}\,\psi_\alpha(x'') = -\Delta x'\frac{\partial}{\partial x''}\psi_\alpha(x'')$$

坐标表象 微分算符. $\boxed{\hat{p} = -i\hbar\frac{\partial}{\partial x}}$