§ Dirac 方程：寻找下定的波函数的解释程
希望写个一阶偏微方程.

$$\boxed{(i\gamma^\mu \partial_\mu - m)\psi(\vec{x},t)=0.}$$

用 $(-i\gamma^\nu\partial_\nu - m)$ 作用到上式，回到 K-G eq.

$$(\gamma^\nu\partial_\nu\gamma^\mu\partial_\mu + m^2)\psi(\vec{x},t)=0$$

只要条件 $\gamma^\nu\partial_\nu\gamma^\mu\partial_\mu = \partial^\mu\partial_\mu = g^{\mu\nu}\partial_\nu\partial_\mu$ 满足.

即：$\frac{1}{2}(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu) \equiv \frac{1}{2}\{\gamma^\mu,\gamma^\nu\} = g^{\mu\nu}$.
（利用了 ν,μ 是哑标：$\gamma^\nu\partial_\nu\gamma^\mu\partial_\mu = \gamma^\mu\partial_\mu\gamma^\nu\partial_\nu$）

- 4个 γ^μ, $\mu=0,1,2,3$ 是矩阵，遵从 Clifford 代数.

$$\begin{cases}(\gamma^0)^2 = 1, \\ (\gamma^i)^2 = -1, \quad i=1,2,3 \\ \gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu, \quad \text{if } \mu\neq\nu\end{cases}$$

把 $\psi(\vec{x},t) = e^{-ip^\mu x_\mu}$ 代入 Dirac Eq.

- 其实是 $\Psi e^{-ip^\mu x_\mu}$（代）
Ψ 有4个分量但与 $(t,\vec{x})$ 无关

- 方程成立条件：$\gamma^\mu p_\mu - m = 0$. $\quad (p^\mu x_\mu = p_\mu x^\mu,\ \partial_\mu = \frac{\partial}{\partial x^\mu})$

即 $\gamma^0(\gamma^0 E - \vec{\gamma}\cdot\vec{p} - m) = 0$

$$\Rightarrow E = \gamma^0\vec{\gamma}\cdot\vec{p} + \gamma^0 m$$

$\gamma^0(i\gamma^\mu\partial_\mu - m)\Psi e^{-ip^\mu x_\mu} = 0$
可理解为 $E\psi(x,t) = \hat{H}\psi(x,t)$
$\psi(x,t) = \psi_E(x)e^{-iEt}$
$\Rightarrow$ 本征方程 $\boxed{\hat{H}\psi_E(x) = E\psi_E(x)}$
- 此为定态 Dirac 方程

- 定义 $\alpha_i \equiv \gamma^0\gamma^i$, $\beta \equiv \gamma^0$

$$H = \vec{\alpha}\cdot\vec{p} + \beta m \longleftarrow \text{Dirac Hamiltonian}$$

由 $i\frac{\partial\psi}{\partial t} = H\psi$ 写出另一种形式的 Dirac 方程：$\boxed{i\frac{\partial\psi(x,t)}{\partial t} = (\vec{\alpha}\cdot\vec{p} + \beta m)\psi(x,t)}$

- 如果考虑 电磁作用. $p^{\mu} \to p^{\mu} - eA^{\mu}$,
  - 当 $\vec{A}=0$, $A_0=\phi$

$$H = \alpha\cdot\vec{p} + \beta m + e\phi$$

决定带电粒子在静电势 ϕ 下的运动.

- 实现 Clifford 代数的矩阵至少是 4×4 的.

我们要求 α, β 是 Hermitian matrices $\Rightarrow \gamma^0 = \gamma^{0\dagger}$,
如果 $\vec{\gamma}$ 反厄米, 即 $\vec{\gamma}^{\dagger} = -\vec{\gamma}$
可保证: $\alpha_i^{\dagger} = \gamma^{i\dagger}\gamma^{0\dagger} = -\gamma^i\gamma^0 = \gamma^0\gamma^i = \alpha_i$

选择: $\beta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{bmatrix}$

$\vec{\alpha} = \begin{bmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{bmatrix}$ ($\vec{\sigma}$ Pauli 矩阵)

$\gamma^1 = \gamma^0\alpha_1 = \begin{bmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{bmatrix}\begin{bmatrix} 0 & \sigma_x \\ \sigma_x & 0 \end{bmatrix} = \begin{bmatrix} 0 & \sigma_x \\ -\sigma_x & 0 \end{bmatrix}$

- 正定几率密度: 设 Ψ 列矢, $\Psi^{\dagger}$ 行矢.

$$\rho = \Psi^{\dagger}\Psi = (\psi_1^*, \psi_2^*, \psi_3^*, \psi_4^*)\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \Rightarrow \boxed{\text{正定!}}$$

ρ 满足连续性方程: $\dfrac{\partial\rho}{\partial t} + \nabla\cdot\vec{j} = 0$.

$$\vec{j} = \Psi^{\dagger}\vec{\alpha}\Psi = (\Psi^{\dagger}\alpha_1\Psi, \Psi^{\dagger}\alpha_2\Psi, \Psi^{\dagger}\alpha_3\Psi)$$

证明: $i\frac{\partial}{\partial t}|\psi\rangle = H|\psi\rangle = (\vec{\alpha}\cdot\hat{\vec{p}} + \beta m)|\psi\rangle$

$$\langle x|i\frac{\partial}{\partial t}|\psi\rangle = \langle x|H|\psi\rangle$$

$$i\frac{\partial}{\partial t}\Psi = -i\vec{\alpha}\cdot\nabla\Psi + \beta m\Psi$$

$$-i\frac{\partial}{\partial t}\Psi^{\dagger} = i\vec{\alpha}\cdot\nabla\Psi^{\dagger} + \beta m\Psi^{\dagger}$$

$$i\frac{\partial}{\partial t}(\Psi^{\dagger}\Psi) = -\nabla\cdot(\Psi^{\dagger}\vec{\alpha}\Psi)$$

但习惯上人们用 $\bar{\Psi} = \Psi^+\beta$ 构造 $\rho, \vec{j}$.

$$\rho = \psi^+\psi = \psi^+\gamma^0\gamma^0\psi = \bar{\Psi}\gamma^0\psi$$

$$\vec{j} = \psi^+\gamma^0\gamma^0\vec{\alpha}\psi = \bar{\Psi}\gamma^0\vec{\alpha}\psi = \bar{\Psi}\vec{\gamma}\psi$$

$$\Rightarrow \frac{\partial}{\partial t}(\bar{\Psi}\gamma^0\psi) + \nabla\cdot(\bar{\Psi}\vec{\gamma}\psi) = \partial_\mu j^\mu = 0.$$

其中. $j^\mu = \bar{\Psi}\gamma^\mu\psi$ 4矢形式

体积元 $d^3x' = \gamma d^3x$ 尺缩效应

$dt' = \frac{dt}{\gamma}$ "钟慢"

$j^0 d^3x$ 不变量

考虑自由粒子: $(\gamma^\mu p_\mu - m)\psi(x,t) = 0$, 其共轭形式 $(p_\mu - m)\Psi^+(\gamma^\mu)^+ = 0$

$$\Rightarrow (p_\mu - m)\bar{\Psi}\gamma_0\gamma^\mu = 0$$

$$j^\mu = \frac{1}{2}\left[(\bar{\Psi}\gamma^\mu)\Psi + \bar{\Psi}(\gamma^\mu\Psi)\right] = \frac{1}{2m}\left[(\bar{\Psi}\gamma^\mu)\gamma^\nu p_\nu\Psi + \bar{\Psi}\gamma^\nu p_\nu(\gamma^\mu\Psi)\right]$$

$$= \frac{1}{2m}\bar{\Psi}\underbrace{(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu)}_{2g^{\mu\nu}}p_\nu\Psi = \frac{p^\mu}{m}\bar{\Psi}\Psi = \begin{cases} j^0 = \frac{E}{m}\bar{\Psi}\Psi = \frac{1}{\gamma}\bar{\Psi}\Psi \\ \vec{j} = \frac{\vec{p}}{m} = \frac{\vec{v}}{\gamma}\bar{\Psi}\Psi \end{cases}$$

$$\gamma = \sqrt{1 - v^2/c^2}$$

- 自由粒子解.

1. 考虑一个动量 $\vec{p} = 0$ 的粒子.

$$\beta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$i\gamma^0\frac{\partial\Psi}{\partial t} = m\Psi \Rightarrow i\frac{\partial\Psi}{\partial t} = \beta m\Psi.$$

4个解: $\underbrace{\Psi_1 = e^{-imt}\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \Psi_2 = e^{-imt}\begin{bmatrix}0\\1\\0\\0\end{bmatrix}}_{\text{spin up 正能量解. spin down}}, \underbrace{\Psi_3 = e^{imt}\begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \Psi_4 = e^{imt}\begin{bmatrix}0\\0\\0\\1\end{bmatrix}}_{\text{up 负能量解. down}}$

2. 考虑. $\vec{p} = p\hat{z}$. 沿z方向运动自由粒子.

求解本征方程 $H\Psi = E\Psi$ with $H = \alpha_z p + \beta m$

$$\alpha_z = \begin{bmatrix} 0 & \sigma_z \\ \sigma_z & 0 \end{bmatrix} \quad \begin{bmatrix} m & p\sigma_z \\ p\sigma_z & -m \end{bmatrix}\begin{bmatrix}u_1\\u_2\\u_3\\u_4\end{bmatrix} = E\begin{bmatrix}u_1\\u_2\\u_3\\u_4\end{bmatrix}$$

u_1 与 u_3 couple, u_2 与 u_4 couple.