§ 海森堡方程.

H picture下, H方程.

$$\frac{dA}{dt} = \frac{1}{i\hbar}[A, H]$$

Erenfest 定理, 以上对应经典方程.

我们来验证一下.

$$[\hat{x}_i, \hat{p}_i] = i\hbar$$

$$\frac{dx_i}{dt} = \frac{1}{i\hbar}[x_i, H], \quad H = \frac{(P-\frac{qA}{c})^2}{2\mu} + q\phi$$

$$[x_i, \phi] = 0. \quad [x_i, A] = 0, \quad \vec{v}^2 = \vec{v}\cdot\vec{v} = \sum_i v_i^2$$

$$\left[x_i, (P-\frac{qA}{c})^2\right] = (P_i - \frac{qA_i}{c})[x_i, P_i - \frac{qA_i}{c}]$$

$$+ [x_i, P_i - \frac{qA_i}{c}](P_i - \frac{qA_i}{c})$$

$$= 2i\hbar(P_i - \frac{qA_i}{c})$$

$$\therefore \frac{dx_i}{dt} = \frac{1}{\mu}(P_i - \frac{qA_i}{c}) = \frac{\pi_i}{\mu}$$

用量子力学得到 $\vec{p}$ 与 $\vec{\pi}$ 的关系, 说明 $\hat{p} = i\hbar\nabla$ 不确

再来验证 $\mu\dot{\vec{v}} = q\vec{E} + q\vec{v}\times\vec{B}$

- 首先计算 $[\pi_i, \pi_j] = \frac{i\hbar q}{c}\varepsilon_{ijk}B_k$

$$\varepsilon_{123}=1,\quad \varepsilon_{213}=-1,\quad \varepsilon_{231}=1\ \cdots$$

$$[\pi_1, \pi_2] = \left[P_1 - \frac{qA_1}{c},\ P_2 - \frac{qA_2}{c}\right]$$

$$= [P_1, P_2] - \left[P_1, \frac{qA_2}{c}\right] - \left[\frac{qA_1}{c}, P_2\right] + \left[\frac{qA_1}{c}, \frac{qA_2}{c}\right]$$

利用：$[P_1, A_2]\psi = -i\hbar\partial_x A_2\psi + A_2 i\hbar\partial_x\psi = (-i\hbar\partial_x A_2)\psi$

$$-\left[\frac{qA_1}{c}, P_2\right]\psi = \left[P_2, \frac{qA_1}{c}\right]\psi = \frac{q}{c}(-i\hbar\partial_y A_1)\psi$$

$$[\pi_1, \pi_2] = \frac{qi\hbar}{c}(\partial_x A_2 - \partial_y A_1) = \frac{qi\hbar}{c}B_3$$

$$\vec{B} = \nabla\times\vec{A}$$

- $[\pi_i, \phi] = \left[P_i - \frac{qA}{c}, \phi\right] = -i\hbar\partial_i\phi$

$$\Rightarrow \frac{d\pi_i}{dt} = \frac{1}{i\hbar}\left[\pi_i, \frac{\pi^2}{2\mu} + \phi\right],\quad \pi^2 = \pi_1^2 + \pi_2^2 + \pi_3^2$$

$$\text{以}\ \frac{d\pi_1}{dt} = \frac{1}{i\hbar}\left(\left[\pi_1, \frac{\pi_2^2}{2\mu}\right] + \left[\pi_1, \frac{\pi_3^2}{2\mu}\right] + [\pi_1, q\phi]\right)$$

然而：$\pi = P - \frac{qA}{c}$，当 $\frac{\partial A(x,t)}{\partial t} \neq 0$，$\frac{d\pi_i}{dt} = \frac{1}{i\hbar}[\pi_i, H] - \frac{q\partial A}{c\partial t}$

$$[\pi_1, \pi_2^2] = \pi_2[\pi_1, \pi_2] + [\pi_1, \pi_2]\pi_2$$

$$= (\pi_2 B_3 + B_3 \pi_2)\frac{i\hbar q}{c}$$

$$[\pi_1, \pi_3^2] = -(\pi_3 B_2 + B_2 \pi_3)\frac{i\hbar q}{c}$$

$$\frac{d\pi_1}{dt} = \frac{1}{i\hbar}\left(-i\hbar \partial_x q\phi + \frac{i\hbar q}{2\mu c}\left[(\pi_2 B_3 - \pi_3 B_2) - (B_2\pi_3 - B_3\pi_2)\right]\right)$$

$$- \frac{q}{c}\frac{\partial A}{\partial t}$$

$$\boxed{\frac{d\vec{\pi}}{dt} = q\vec{E} + \frac{q}{c \cdot 2\mu}(\vec{\pi}\times\vec{B} - \vec{B}\times\vec{\pi})}$$

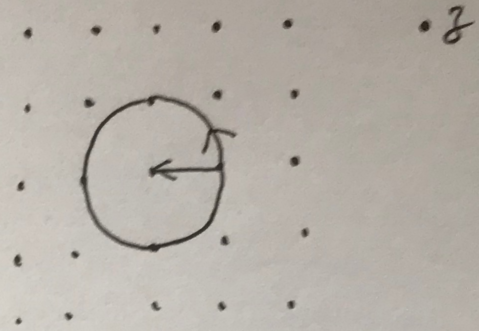
$$\because \vec{E} = -\nabla\phi - \frac{\partial\vec{A}}{c\partial t}, \quad \frac{\vec{\pi}}{\mu} = \frac{d\vec{x}}{dt},$$

$$\vec{\pi}\times\vec{B} = \begin{vmatrix} i & j & k \\ \pi_1 & \pi_2 & \pi_3 \\ B_1 & B_2 & B_3 \end{vmatrix}, \quad \vec{B}\times\vec{\pi} = \begin{vmatrix} i & j & k \\ B_1 & B_2 & B_3 \\ \pi_1 & \pi_2 & \pi_3 \end{vmatrix}$$

- 验证了 Erenfest 定理.
- $\hat{p} = -i\hbar\nabla$ 正确.

§朗道能级 (Landau Level)

电子在均匀磁场中



· 经典

$\vec{v}$ 圆运动.

$$\frac{eBv}{c} = m\frac{v^2}{r}$$

解出半径 $r = \frac{mc}{eB}\cdot v$

周期 $\frac{2\pi r}{v} = \frac{2\pi mc}{eB} = T$

圆频率 $\omega_c = \frac{2\pi}{T} = \frac{eB}{mc}$

· 量子力学.

$B = \nabla\times A$.

A 有两种选法.

$\vec{B} = B\hat{z}$

1. $\vec{A} = \frac{1}{2}\vec{B}\times\vec{r}$
2. $A_x = -By,\ A_y = A_z = 0$

For 1. $A_x = -\frac{1}{2}BY,\quad A_y = \frac{Bx}{2},\quad A_z = 0$

$$\because \vec{A} = \frac{1}{2}\begin{vmatrix} i & j & k \\ 0 & 0 & B \\ x & y & z \end{vmatrix},\qquad \boxed{\nabla\cdot\vec{A} = 0.}$$

$$\nabla\times\vec{A} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ -\frac{BY}{2} & \frac{Bx}{2} & 0 \end{vmatrix} = B\hat{z}$$

· 对称规范.

For 2. $$\nabla\times\vec{A} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ -BY & 0 & 0 \end{vmatrix} = B\hat{z}.$$

· Landau gauge

$$\boxed{\nabla\cdot\vec{A} = 0}$$

首先在 Landau Gauge 下讨论.

$$H=\frac{1}{2\mu}\left[\left(P_x-\frac{eB}{c}y\right)^2+P_y^2+P_z^2\right]$$

- z 方向特殊. 设 $\psi(x,y,z)=\psi(x,y)e^{ik_z z}$

$[P_z, H]=0$

$$P_z=\hbar k_z \quad \text{好量子数.}$$

$$E_z=\frac{\hbar^2 k_z^2}{2\mu}$$

$$H=H_{xy}+H_z$$

$$=\frac{1}{2\mu}\left[\left(P_x-\frac{eB}{c}y\right)^2+P_y^2\right]+\frac{P_z^2}{2\mu}$$

① $\boxed{H_{xy}\psi_{E_{xy}}(x,y)=E_{xy}\psi_{E_{xy}}(x,y)}$, $E=E_{xy}+E_z$

- H_{xy} 中没有 x 分量坐标 $\Rightarrow [P_x, H_{xy}]=0$

完全集 (P_x, H) (以下 E_{xy} 简写为 E)

设 $\psi_E(x,y)=e^{ik_x x}\phi(y)$

$\because \hat{P}_x e^{ik_x x}=\hbar k_x e^{ik_x x}$

$\hbar k_x$ 是 P_x 的本征值, 记为 P_x'

代入①.

$$\frac{1}{2\mu}\left[\left(P_x'-\frac{eB}{c}y\right)^2+\hat{P}_y^2\right]\phi(y)=E\phi(y)$$

令 $y_0 = \frac{cP_x'}{eB}$，　$\omega_c = \frac{eB}{\mu c}$

$$\frac{1}{2\mu}\left[(y_0-y)^2\frac{e^2B^2}{c^2} - \frac{\hbar^2 d^2}{dx^2}\right]\phi(y) = E\phi(y)$$

$$-\frac{\hbar^2}{2\mu}\phi''(y) + \frac{1}{2}\mu\omega_c^2(y-y_0)^2\phi(y) = E\phi(y)$$

• 此为一维谐振子，中心位置为 y_0，频率 ω_c

1. ∴ $E_n = (n+\frac{1}{2})\hbar\omega_c$，　$n = 0, 1, 2, \cdots$

著名的 Landau Level.

2. 本征函数：

$$\phi_n(y) = N_n e^{-\frac{\alpha^2(y-y_0)^2}{2}} H_n(\alpha(y-y_0))$$

$$\alpha = \sqrt{\frac{\mu\omega_c}{\hbar}} = \frac{1}{x_0}$$

$$\psi_E(x,y) = e^{ik_x x}\phi_n(y)$$

$\underline{\frac{cP_x'}{eB} = y_0 \text{ 由 } k_x \text{ 决定}}$ $\Rightarrow$ 大量简并！

3. 假设 $L_x \times L_y$ 周期边界系统

$$e^{ik_x(L_x+x)} = e^{ik_x x} \Rightarrow k_x L_x = 2\pi \cdot n_x, \quad n_x = 0, \pm 1, \cdots$$

y_0 的取值 $0<y_0\le L_y$，即

$$0<y_0=\frac{c\hbar 2\pi}{eBL_x}\times n_x\le L_y.$$

n_x 可从 0 取到 $N=\frac{L_xL_y\,eB}{hc}$

$$=\frac{AB}{\left(\frac{hc}{e}\right)}$$

此即 Landau Level 的简并度：AB 总磁通

$\frac{hc}{e}$ 也是磁通量纲

称磁通量子

4. 对比经典图像.

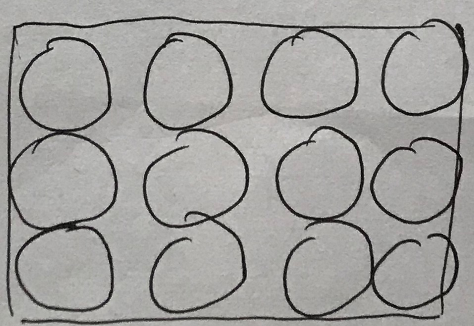
量子基态 $E_0=\frac{1}{2}\hbar\omega_c \longrightarrow \frac{1}{2}\mu v_0^2=E_0$

$$v_0^2=\frac{\hbar\omega_c}{\mu}$$

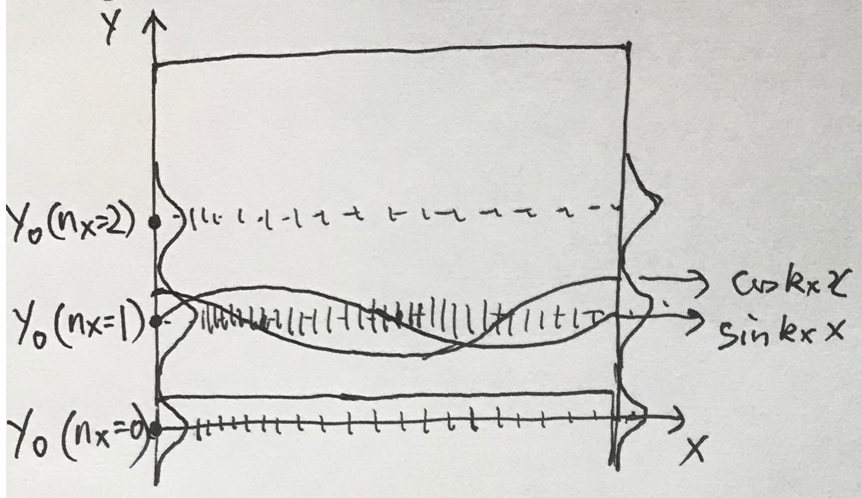
半径 $r_0=\frac{v}{\omega_c}\Rightarrow \pi r_0^2=\frac{h}{2\mu\omega_c}=\frac{hc}{2eB}$

$$\omega_c=\frac{eB}{\mu c}$$

即 ○ 磁通 $=\frac{1}{2}\frac{hc}{e}$



5. 波函数的图像，基态 $E_0=\frac{1}{2}\hbar\omega_c$



$k_x=\frac{2\pi}{L}$

$k_x=0$.

几率密度：$\rho=|\psi|^2=\phi_0^2(y)$，x方向均匀，y方向以 y_0 为中心

几率流密度：$\vec{j}=\frac{\rho}{\mu}\left(\nabla S-\frac{q\vec{A}}{c}\right)$

$$e^{ik_x x}\longrightarrow S=\hbar k_x x;\quad \vec{A}=-By\vec{e}_x$$

$$\nabla\cdot\vec{j}=\frac{\nabla\rho}{\mu}\cdot\left(\nabla S-\frac{q\vec{A}}{c}\right)+\frac{\rho}{\mu}\left(\nabla\cdot\nabla S-\frac{q}{c}\nabla\cdot\vec{A}\right)$$

由于 $\nabla S=\hbar k_x\vec{e}_x$，$\nabla\cdot\nabla S=0$

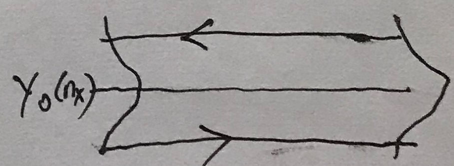
$$\nabla\cdot\vec{A}=\partial_x A_x+\partial_y A_y+\partial_z A_z=0$$

$$\rho(y)\Rightarrow\partial_x\rho=0\Rightarrow\nabla\rho\cdot\nabla S=0$$

$$\nabla\rho\cdot\vec{A}=0$$

$\therefore\ \nabla\cdot\vec{j}=0$.

$\therefore\ \frac{\partial\rho}{\partial t}=0$.



$$\vec{j}=\frac{\rho}{\mu}\left(\hbar k_x-\frac{eB}{c}y\right)\vec{e}_x$$

考虑 $y=y_0(n_x)$，$\vec{j}=\frac{\rho}{\mu}\left(\hbar k_x-\frac{eB}{c}\cdot\frac{c\hbar k_x}{eB}\right)=0$