规范变换与规范不变

规范变换（gauge transform)

$$\overrightarrow{A} \rightarrow \overrightarrow{A}' = \overrightarrow{A} + \nabla\Lambda(\mathbf{r}, t)$$

$$\psi \rightarrow \phi' = \phi - \frac{\partial\Lambda(\mathbf{r}, t)}{c\partial t}$$

- 保证E，B不变

$$i\hbar\frac{\partial\psi'}{\partial t} = [\frac{1}{2\mu}(-i\hbar\nabla - \frac{q}{c}\overrightarrow{A}')^2 + q\phi']\psi'$$

$$\psi \rightarrow \psi' = e^{i\frac{q\Lambda}{\hbar c}}\psi$$

- 证明

$$i\hbar\frac{\partial}{\partial t}(e^{\frac{iq\Lambda}{\hbar c}}\psi) = -\frac{q\partial\Lambda}{c\partial t}\psi' + e^{\frac{iq\Lambda}{\hbar c}}i\hbar\frac{\partial}{\partial t}\psi$$

$$(-i\hbar\nabla - \frac{q}{c}\overrightarrow{A}')\psi' = e^{\frac{iq\Lambda}{\hbar c}}(-i\hbar\nabla - \frac{q}{c}\overrightarrow{A})\psi$$

- 不改变波函数的几率解释
  1. ρ不变
  2. j 不变

$$j' = \frac{\rho}{\mu}(\nabla(S + \frac{q\Lambda}{c}) - \frac{q}{c}(A + \nabla\Lambda)) = j$$

- 多出的相因子是位置和时间的函数，但是容易证明坐标和机械动量的期望值不变

$$\vec{j} = \frac{1}{2\mu}(\psi^*(P - \frac{qA}{c})\psi + \psi(P - \frac{qA}{c})^*\psi^*)$$

$$\int d^3x\vec{j} = \frac{\langle(P - \frac{qA}{c})\rangle}{\mu} = \frac{\langle\Pi\rangle}{\mu}$$

规范不变 （gauge invariant)

更一般描述

$$|\alpha'\rangle = \mathscr{G}|\alpha\rangle$$

要求

$$\langle\alpha|\alpha\rangle = \langle\alpha'|\alpha'\rangle = \langle\alpha|\mathscr{G}^\dagger\mathscr{G}|\alpha\rangle = 1$$

- Hilbert 空间转动
- 幺正变换 $\mathscr{G}^\dagger \cdot \mathscr{G} = 1$

规范不变要求： $\langle\alpha|\mathscr{G}^\dagger\mathbf{x}\mathscr{G}|\alpha\rangle = \langle\alpha|\mathbf{x}|\alpha\rangle \to [\mathscr{G}, x] = 0$

$$\langle\alpha|\mathscr{G}^\dagger\mathbf{\Pi}'\mathscr{G}|\alpha\rangle = \langle\alpha|\mathbf{\Pi}|\alpha\rangle$$

$$\mathscr{G}^\dagger(\mathbf{\Pi} - \frac{q\nabla\Lambda}{c})\mathscr{G} = \mathbf{\Pi} \to [P, \mathscr{G}] = \frac{q\nabla\Lambda}{c}\mathscr{G}$$

坐标表象下： $\langle x|\mathscr{G}|\alpha\rangle = \psi_{\mathscr{G}\alpha}(x) = \mathscr{G}(x)\psi_\alpha(x)$

$$-i\hbar\nabla\mathscr{G}(x) = \frac{q\nabla\Lambda}{c}\mathscr{G}(x) \to \mathscr{G}(x) = e^{\frac{iq\Lambda}{\hbar c}}$$

Heisenberg方程

$$\frac{dx_i}{dt} = \frac{1}{i\hbar}[x_i, H] = \frac{1}{\mu}(P_i - \frac{qA_i}{c})$$

- 验证机械动量的物理意义

$$[\Pi_i, \Pi_j] = \frac{i\hbar q}{c}\epsilon_{ijk}B_k$$

$$[\Pi, \phi] = [P, \phi] = -i\hbar\nabla\phi$$

得到Π满足的运动方程

$$\mu\frac{d^2\overrightarrow{x}}{dt^2} = \frac{d\Pi}{dt} = q[\overrightarrow{E} + \frac{1}{2c}(\frac{d\overrightarrow{x}}{dt} \times \overrightarrow{B} - \overrightarrow{B} \times \frac{d\overrightarrow{x}}{dt})]$$

- 与经典力学的牛顿方程形式一致