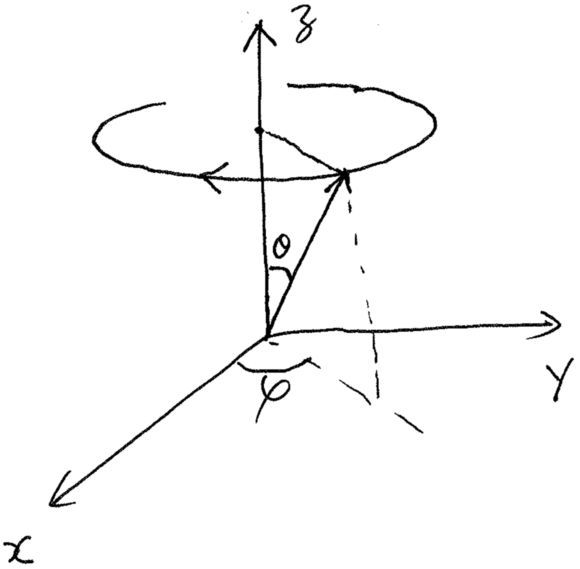
我们再来复习一个有趣的问题：Larmor pression（进动）



固定磁场 $\vec{B} = B\hat{z}$，

中子磁矩 $\vec{\mu} = g\mu_0 \vec{\sigma}$

$$H = -\vec{\mu}\cdot\vec{B} = -\frac{\hbar\omega}{2}\sigma_z = -\omega S_z$$

$$\mu_0 = \frac{g_n e\hbar}{2mc},\quad \mu_0 B = \frac{\hbar\omega}{2}$$

$$\omega = \frac{g_n e B}{mc}$$

初态 $|t=0\rangle = |\vec{n}(0)\rangle = \begin{pmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}e^{i\varphi}\end{pmatrix} = \begin{pmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}\end{pmatrix} \leftarrow \varphi=0.$

末态 $t=t'$：$|t'\rangle = e^{-\frac{iHt'}{\hbar}}|\vec{n}(0)\rangle$

$$\boxed{\begin{aligned}&\text{经典：}\frac{d\vec{S}}{dt} = \vec{\mu}\times\vec{B}\\ &\vec{S}\text{ 只改变方向：}|S|\frac{d\phi}{dt}\sin\theta = \mu_0 B\sin\theta\end{aligned}}$$

$U(t', t=0) = e^{-i\frac{Ht'}{\hbar}} = e^{-\frac{iS_z}{\hbar}\cdot\phi}$

$$= e^{-i\frac{Ht'}{\hbar}}\left(\cos\frac{\theta}{2}|+\rangle + \sin\frac{\theta}{2}|-\rangle\right)$$

$$= \cos\frac{\theta}{2}e^{\frac{i\omega t'}{2}}|+\rangle + \sin\frac{\theta}{2}e^{-\frac{i\omega t'}{2}}|-\rangle$$

$$\frac{d\phi}{dt} = \omega$$

$$\phi(t') = \phi(0) + \omega t'$$

其中 $\phi = -\omega t'$

即绕 z 轴转动 $-\omega t'$

$= D_z(\phi = -\omega t')$

$$= \begin{pmatrix}\cos\frac{\theta}{2}e^{\frac{i\omega t'}{2}}\\ \sin\frac{\theta}{2}e^{-\frac{i\omega t'}{2}}\end{pmatrix}$$

$$\boxed{\begin{aligned}&\text{Heisenberg Eq：}\frac{dS_z}{dt} = \frac{1}{i\hbar}[S_z, H] = 0\\ &\frac{dS_x}{dt} = \omega S_y,\ \frac{dS_y}{dt} = -\omega S_x\end{aligned}}$$

计算 $\vec{\sigma}$ 的期望值 $\langle t'|\vec{\sigma}|t'\rangle = (\langle t'|\sigma_x|t'\rangle, \langle t'|\sigma_y|t'\rangle, \langle t'|\sigma_z|t'\rangle)$

注意到：$|t'\rangle = \begin{pmatrix}\cos\frac{\theta}{2}e^{-\frac{i\varphi(t')}{2}}\\ \sin\frac{\theta}{2}e^{\frac{i\varphi(t')}{2}}\end{pmatrix} = |\vec{n}(\theta,\varphi)\rangle$，$\varphi(t') = -\omega t'$

$\therefore \langle t'|\vec{\sigma}|t'\rangle = \vec{n}(\theta,\varphi) = (\sin\theta\cos\omega t', \sin\theta\sin(-\omega t'), \cos\theta)$

这就是自旋角动量的 Larmor pression．周期为 $T = \frac{2\pi}{\omega}$

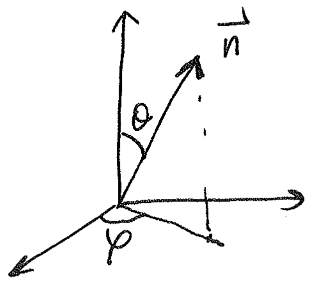
但是 $|t'\rangle$ 的周期为 $T' = \frac{4\pi}{\omega}$！

第十四讲 → 第十六讲 (2022)

§ 自旋系统的路径积分

我们考虑一个自旋量子数 $I=\frac{1}{2}$ 的粒子.（S 为作用量保留）.

$$|\psi\rangle = \alpha|\uparrow\rangle + \beta|\downarrow\rangle = \begin{pmatrix}\alpha\\ \beta\end{pmatrix}$$



其中 $|\alpha|^2+|\beta|^2=1$. 也可以写成

$$|\psi\rangle = e^{ib}\left(\cos\frac{\theta}{2}\, e^{-\frac{i\varphi}{2}}|\uparrow\rangle + e^{\frac{i\varphi}{2}}\sin\frac{\theta}{2}|\downarrow\rangle\right) = |b,\theta,\varphi\rangle$$

是 $\sigma_n = \vec{\sigma}\cdot\vec{n}$，$\vec{n}=(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$ 的本征值为 1 的本征态. e^{ib} 是任意的相因子，或gauge变换自由度.

- 我们看到 $|b,\theta,\varphi\rangle$ 是完备的.（b 固定，对 θ,φ 积分）.

$$\frac{1}{2\pi}\int_0^{\pi}\sin\theta d\theta \underbrace{\int_0^{2\pi} d\varphi}_{\text{积分写成 }\frac{1}{2\pi}\int d\vec{n}} |b,\theta,\varphi\rangle\langle b,\theta,\varphi| = |\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow| = \mathbb{1}$$

- $\vec{I}=\frac{\hbar}{2}\vec{\sigma}$ 的期望值 就是 $\frac{\vec{n}\hbar}{2}$ 或 $(\langle\vec{\sigma}\rangle=\vec{n})$

$$\langle b,\theta,\varphi|\frac{\hbar\vec{\sigma}}{2}|b,\theta,\varphi\rangle \overset{\text{固定}b}{=} \frac{1}{2}\langle\vec{n}|\vec{\sigma}|\vec{n}\rangle = \frac{\vec{n}}{2} \quad (\text{取}\hbar=1)$$

- 但是 不同的 $|b,\theta,\varphi\rangle$ 不正交.

$$\langle\vec{n}|\vec{n}'\rangle \neq \delta(\vec{n}-\vec{n}')$$

为了使用路径积分工具，有完备性就行. 这个表象称自旋相干态表象.

- 可推广到任意 I：$(2I+1)\int\frac{d^2n}{4\pi}|\vec{n}\rangle\langle\vec{n}| = \mathbb{1}$，$\left[\int d^2 n = \int_0^{\pi}\sin\theta d\theta\int_0^{2\pi}d\varphi\right]$

$|S,S_z\rangle$，$S_z=-S,\cdots,0,\cdots S$. $\quad\langle\vec{n}|\vec{S}|\vec{n}\rangle = I\vec{n}$ ①

$$|\langle\vec{n}_1|\vec{n}_2\rangle|^2 = \left(\frac{1+\vec{n}_1\cdot\vec{n}_2}{2}\right)^{2I}$$

我们来计算从$|\vec{n}\rangle$出发到$|\vec{n}'\rangle$的传播子 $(\hbar=1)$

$$\underset{t=t'}{\langle\vec{n}'|}e^{-iHt'}\underset{t=0}{|\vec{n}\rangle}=\int\prod_{i=1}^{N-1}\frac{d\vec{n}_i}{2\pi}\langle\vec{n}'|e^{-i\Delta tH}|\vec{n}_{N-1}\rangle\langle\vec{n}_{N-1}|e^{-i\Delta tH}|\vec{n}_{N-2}\rangle\cdots$$

$$\cdots\cdots\times\langle\vec{n}_{k+1}|e^{-i\Delta tH}|\vec{n}_k\rangle\cdots\langle\vec{n}_1|e^{-i\Delta tH}|\vec{n}\rangle$$

初时刻为$t_0=0$，末时刻$t_N=t'$，$\dfrac{t'-t}{N}=\Delta t$. 以$I=\frac{1}{2}$粒子为例.

$|\vec{n}_k\rangle$是t_k时刻的态矢，$t_k=k\cdot\Delta t$

引入简写 $|t\rangle=|\vec{n}(t)\rangle=|b(t),\theta(t),\varphi(t)\rangle$

① $\langle t+\Delta t|e^{-i\Delta tH}|t\rangle\simeq\langle t+\Delta t|(1-i\Delta tH)|t\rangle$

$$=\langle t+\Delta t|t\rangle-i\Delta t\langle t+\Delta t|H|t\rangle$$

由于 $|t+\Delta t\rangle=|t\rangle+\Delta t\dfrac{d|t\rangle}{dt}$ $\leftarrow d|t\rangle\equiv|t+\Delta t\rangle-|t\rangle$

简写 $=|t\rangle+\Delta t\,|\dot{t}\rangle$

$\therefore$ ① $=(\langle t|+\Delta t\langle\dot{t}|)\cdot|t\rangle-i\Delta t\langle t|H|t\rangle$ $\leftarrow$ 略去Δt^2项

$$=1+\Delta t[\langle\dot{t}|t\rangle-i\langle t|H|t\rangle]$$

$$=e^{-i\Delta t\langle t|H|t\rangle}+\Delta t\langle\dot{t}|t\rangle$$

由于 $\langle t|t\rangle=1\Rightarrow\dfrac{d\langle t|t\rangle}{dt}=\langle\dot{t}|t\rangle+\langle t|\dot{t}\rangle=0$

$$=2\mathrm{Re}\langle\dot{t}|t\rangle$$

$\therefore\langle\dot{t}|t\rangle$是纯虚数 $=-\langle t|\dot{t}\rangle$

① $=e^{-i\Delta t\langle t|H|t\rangle}+i\Delta t\cdot i\langle t|\dot{t}\rangle$

$\Delta t\cdot i\langle t|\dot{t}\rangle$是相位，反映$|t\rangle$与$|t+\Delta t\rangle$的"连络"

$$\langle \vec{n}' | e^{-iHt'} | \vec{n} \rangle = \int \prod_{k=1}^{N-1} \frac{d^2 n_k}{2\pi} \; e^{i\sum_{k=0}^{N-1} \Delta t [i\langle t_k | \dot{t}_k \rangle - \langle t_k | H | t_k \rangle]}$$

其中 $\langle t_k | H | t_k \rangle$ 其实为 $\langle \vec{n}(t_k) | H | \vec{n}(t_k) \rangle = H(I\vec{n})$

比如 $H = H(\vec{S}) = -\mu_0 \vec{B} \cdot \vec{S}$，则

$$\langle \vec{n}(t_k) | H(\vec{S}) | \vec{n}(t_k) \rangle = -\mu_0 \vec{B} \cdot [I \vec{n}(t_k)] = H(I\vec{n})$$

最终：

$$\langle \vec{n}' | e^{-iHt'} | \vec{n} \rangle = \int \mathcal{D}\vec{n}(t) \; e^{iS(t)}$$

$$S = \int_0^{t'} dt [i \langle \vec{n}(t) | \dot{\vec{n}}(t) \rangle - H(I\vec{n}(t))]$$

- 我们算的是什么？

$e^{-i\frac{Ht'}{\hbar}} | \vec{n}(0) \rangle = | t' \rangle$，　$\langle \vec{n}' | e^{-i\frac{Ht'}{\hbar}} | \vec{n}(0) \rangle$ 是 t' 时刻自旋处于 $|\vec{n}'\rangle$ 态的几率幅，或传播子.

之前的计算(求解 Seq)告诉我们：$|t'\rangle = |\theta, \varphi = -\omega t'\rangle e^{\frac{i\omega t'}{2}}$.

$\sim |\vec{n}(\theta, \varphi)\rangle$

由于 $\langle \vec{n} | \vec{n}' \rangle$ 没有正交性，　$\langle \vec{n}' | t' \rangle \neq 0$. 即"方向不确定".

(2023 version)

上节课 利用自旋相干态表象 我们计算了传播子

$$\langle \vec{n}(t')| e^{-\frac{iH}{\hbar}(t'-t)} |\vec{n}(t)\rangle = \int \mathcal{D}\vec{n}(\tau)\ e^{i\frac{S}{\hbar}}$$

其中 $\int \mathcal{D}\vec{n}(\tau) \equiv \int \prod_{k=1}^{N+1} \frac{d\vec{n}}{2\pi}$, $\int d^2\vec{n} = \int d\varphi \int \sin\theta d\theta$

$$S = \int_t^{t'} d\tau \left[i\hbar \langle \vec{n}(\tau) | \dot{\vec{n}}(\tau)\rangle - H(s\vec{n}(\tau)) \right]$$

[ ] 类似 $L = \cancel{\ } p\dot{x} - H(x,p)$, $\int_t^{t'} d\tau \langle \vec{n}(\tau)|\dot{\vec{n}}(\tau)\rangle$ 对相位 ~~纯相位，也称 Berry phase~~

研究一个外磁场中的自旋

$H = -\vec{\mu}\cdot\vec{B}$ $\vec{\mu}$ 为自旋磁矩，设 $\vec{\mu} = \mu_0 \vec{\sigma}$

$\vec{\sigma}$ 为泡利算符.

那么 $H = -\mu_0 \vec{B}\cdot\vec{\sigma}$，利用 $\langle \vec{n}|\vec{\sigma}|\vec{n}\rangle = \vec{n}$

路径积分中 $H(s\vec{n}(\tau)) = -\mu_0 \vec{B}\cdot\vec{n}(\tau)$

下面计算 $\langle \vec{n}(\tau)|\dot{\vec{n}}(\tau)\rangle$， 这里 $|\vec{n}(\tau)\rangle$ 不是 $e^{-iH\tau}|\vec{n}(t=0)\rangle$ 而是一条可能路径上的态

$$|\vec{n}(\tau)\rangle = e^{ib(\tau)} \left[\cos\frac{\theta(\tau)}{2} e^{-\frac{i\varphi(\tau)}{2}} |\uparrow\rangle + \sin\frac{\theta(\tau)}{2} e^{\frac{i\varphi(\tau)}{2}} |\downarrow\rangle \right]$$

$$\text{spinor 表示} = e^{ib(\tau)} \begin{pmatrix} \cos\frac{\theta(\tau)}{2} e^{-\frac{i\varphi(\tau)}{2}} \\ \sin\frac{\theta(\tau)}{2} e^{\frac{i\varphi(\tau)}{2}} \end{pmatrix}$$

b 是任意相位（自由选取）

$$|\dot{\vec{n}}(\tau)\rangle = i\dot{b}(\tau)|\vec{n}(\tau)\rangle + e^{ib(\tau)}\Big[\Big(-\frac{i\dot{\varphi}(\tau)}{2}\cos\frac{\theta(\tau)}{2} - \frac{\dot{\theta}(\tau)}{2}\sin\frac{\theta(\tau)}{2}\Big)e^{-\frac{i\varphi(\tau)}{2}}|\uparrow\rangle$$
$$+\Big(\frac{i\dot{\varphi}(\tau)}{2}\sin\frac{\theta(\tau)}{2} + \frac{\dot{\theta}(\tau)}{2}\cos\frac{\theta(\tau)}{2}\Big)e^{\frac{i\varphi(\tau)}{2}}|\downarrow\rangle\Big]$$

$\theta(\tau)$与$\varphi(\tau)$描述了$\vec{n}$的路径

$$\langle\vec{n}(\tau)|\dot{\vec{n}}(\tau)\rangle = i\dot{b}(\tau) + \cos\frac{\theta(\tau)}{2}\Big(-\frac{i\dot{\varphi}(\tau)}{2}\cos\frac{\theta(\tau)}{2} - \frac{\dot{\theta}(\tau)}{2}\sin\frac{\theta(\tau)}{2}\Big)$$
$$+\sin\frac{\theta(\tau)}{2}\Big(\frac{i\dot{\varphi}(\tau)}{2}\sin\frac{\theta(\tau)}{2} + \frac{\dot{\theta}(\tau)}{2}\cos\frac{\theta(\tau)}{2}\Big)$$

$$= i\Big(\dot{b}(\tau) - \frac{\dot{\varphi}(\tau)}{2}\cos\theta(\tau)\Big)$$

$$S = i\int_t^{t'} d\tau\, i\Big(\dot{b} - \frac{\dot{\varphi}}{2}\cos\theta\Big) - \int_t^{t'} d\tau\, H(S\vec{n}(\tau)) \qquad (\hbar = 1)$$

$$= -\int_t^{t'} d\tau\Big(\dot{b} - \frac{\dot{\varphi}}{2}\cos\theta\Big) - \int_t^{t'} d\tau\, H(S\vec{n}(\tau))$$

下面证明最小作用量原理给出自旋角动量在磁场中的运动方程. $H = -\vec{\mu}\cdot\vec{B} = -\omega\,(\vec{S}\cdot\hat{z}) \leftarrow \vec{B}$指$\hat{z}$

一般指向: $H = -\mu_0\vec{\sigma}\cdot\vec{B}$

$\vec{\sigma}\rightarrow\vec{n}$

首先计算 $\delta\Big[\int_t^{t'} d\tau\, H(S\vec{n}(\tau))\Big]$,

$$= \int_t^{t'} d\tau\, \delta(H(S\vec{n}(\tau))) = \int_t^{t'} d\tau\, \delta\vec{n}(\tau)\cdot\frac{\partial H(S\vec{n}(\tau))}{\partial\vec{n}(\tau)}$$

其中 $\dfrac{\partial H}{\partial\vec{n}} = \Big(\dfrac{\partial H}{\partial n_x}, \dfrac{\partial H}{\partial n_y}, \dfrac{\partial H}{\partial n_z}\Big) = -\mu_0\vec{B}$

因此 $\displaystyle\int_t^{t'} d\tau\, \delta H = -\mu_0\int_t^{t'} d\tau\, \delta\vec{n}\cdot\vec{B}$

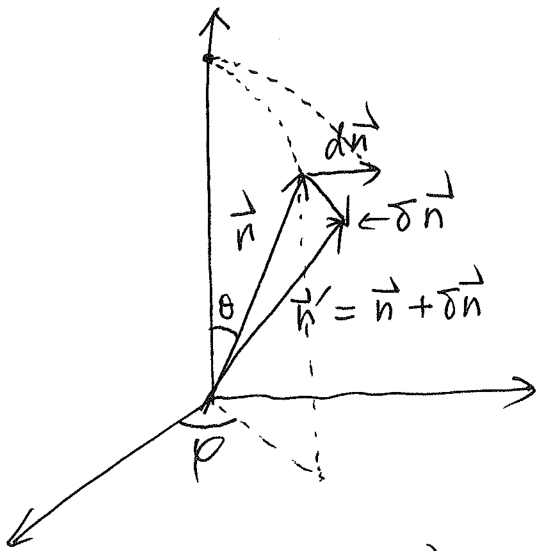
设 $\dot{b}=0$. (一种选择) (也可试一下 $b=\frac{\varphi(\tau)}{2}$)

$$+\int_t^{t'} d\tau\, \delta\left(\frac{\dot\varphi}{2}\cos\theta\right) = \frac{1}{2}\int_t^{t'} d\tau \left[\frac{d\varphi}{d\tau}(-\sin\theta)\delta\theta + \delta\left(\frac{d\varphi}{d\tau}\right)\cos\theta\right]$$

① $$= \frac{1}{2}\int_t^{t'} d\tau\left[-\frac{d\varphi}{d\tau}\sin\theta\,\delta\theta + \frac{d\theta}{d\tau}\sin\theta\,\delta\varphi\right]$$ ← 分部积分

$$\int_t^{t'} d\tau\, \delta\left(\frac{d\varphi}{d\tau}\right)\cos\theta = \int_t^{t'} d\tau \frac{d}{d\tau}(\delta\varphi\cos\theta) + \int_t^{t'} d\tau\, \delta\varphi\sin\theta\frac{d\theta}{d\tau}$$

$$= -\frac{1}{2}\int_t^{t'} d\tau\, \delta\vec{n}\cdot\left(\frac{d\vec{n}}{d\tau}\times\vec{n}\right)$$



$\delta\vec{n}$ 是相邻两条路径 $\vec{n}(\tau)$ 与 $\vec{n}'(\tau)$ 之差

$d\vec{n}$ 是路径 $\vec{n}(\tau+d\tau)$ 与 $\vec{n}(\tau)$ 相邻时刻之差. $\vec{n}(\tau+d\tau)=\vec{n}(\tau)+d\vec{n}$

令 $d\vec{n} = d\theta\,\hat\theta + (\sin\theta\, d\varphi)\,\hat\varphi$

$\delta\vec{n} = \delta\theta\,\hat\theta + (\sin\theta\,\delta\varphi)\,\hat\varphi$

因此 $d\vec{n}\times\vec{n} = -d\theta\,\hat\varphi + \sin\theta\, d\varphi\,\hat\theta$, ∵ $\hat\theta\times\hat{n} = -\hat\varphi$, $\hat\varphi\times\hat{n}=\hat\theta$

$$\frac{d\vec{n}}{d\tau}\times\vec{n} = -\frac{d\theta}{d\tau}\hat\varphi + \sin\theta\frac{d\varphi}{d\tau}\hat\theta$$

$$\delta\vec{n}\cdot\left(\frac{d\vec{n}}{d\tau}\times\vec{n}\right) = \left[\frac{d\varphi}{d\tau}\delta\theta - \frac{d\theta}{d\tau}\delta\varphi\right]\sin\theta$$

最后 得到 ① 式

$$\therefore \delta S = 0 \Rightarrow -\frac{1}{2}\int_t^{t'} d\tau\, \delta\vec{n}\cdot\left(\frac{d\vec{n}}{d\tau}\times\vec{n}\right) + \int_t^{t'} \mu_0\,\delta\vec{n}\cdot\vec{B}\, d\tau = 0$$

$$\Rightarrow \frac{1}{2}\frac{d\vec{n}}{d\tau} \times \vec{n} = \mu_0 \vec{B}$$

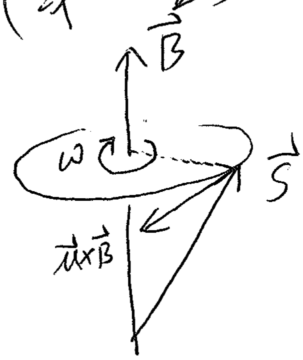
$$\vec{n} \times \left(\frac{1}{2}\frac{d\vec{n}}{d\tau} \times \vec{n}\right) = \mu_0\, \vec{n} \times \vec{B}$$ 注意 $\mu_0 \vec{n}$ 对应 $\mu_0 \vec{\sigma}$ 磁矩

利用 $\vec{c} \times (\vec{a} \times \vec{b}) = (\vec{c}\cdot\vec{b})\vec{a} - (\vec{c}\cdot\vec{a})\vec{b}$，注意到 $\vec{n}\cdot d\vec{n}=0$
$\vec{n}\cdot\vec{n}=1$

$$\Rightarrow \frac{1}{2}\frac{d\vec{n}}{d\tau} = \mu_0 \vec{n} \times \vec{B}$$

也就是 $\boxed{\dfrac{d\vec{S}}{d\tau} = \vec{\mu} \times \vec{B}}$ $\longrightarrow$ 经典力矩方程.

(或量子 Heisenberg 方程) 给出 $\vec{S}$ 绕 $\vec{B}$ 的进动.



$\omega = 2\mu_0 B$.