第十一讲

上节讲到反铁磁 $H = J\sum_{i=1}^{L} \vec S_i\cdot\vec S_{i+1} - \sum_{i=1}^{L}\vec h_i\cdot\vec S_i$, $J>0$.

$$Z=\int \mathcal{D}\vec n(\tau)\, e^{-S(\tau)} \qquad S(\tau) = 4\pi i S\sum_i W_0(\vec n_i(\tau)) + \int_0^\beta d\tau\, H(S\vec n_1, S\vec n_2,\cdots,S\vec n_L)$$

$\vec n_i(\tau)\to\vec n(x,\tau)$. 步骤. $\vec m(x,\tau)$, $\vec l(x,\tau)$ $\Rightarrow$ $\vec n_i(\tau) = (-1)^i \vec m(x,\tau)+\frac{a}{S}\vec l(x,\tau)$

$\vec n(x,\tau)\, b = \vec m(x,\tau)$, $b^2 = 1-\frac{a^2}{S^2}l^2$

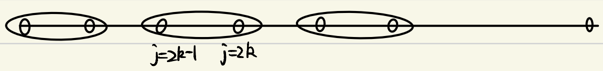
定义 $\rho_s = JS^2 a$, $\frac{1}{\chi_\perp}=4Ja$. $H=\frac12\int dx\left[\rho_s(\partial_x\vec n)^2+\frac{l^2}{\chi_\perp}\right]$

下面我们来看作用量 S 中的 Berry phase 部分: $4\pi i S\sum_i W_0(\vec n_i) \stackrel{记作}{=} S_B$

$$W_0(\vec n_j)=\frac{1}{4\pi}\int_0^\beta d\tau\int_0^1 du\; \vec n_j(u,\tau)\cdot \frac{\partial \vec n_j(u,\tau)}{\partial u}\times\frac{\partial\vec n_j(u,\tau)}{\partial\tau}$$

$$\vec n_i(\tau) = (-1)^i\vec n(x,\tau)\, b + \frac{a}{S}\vec l(x,\tau), \qquad b=\sqrt{1-\frac{a^2l^2}{S^2}}, \qquad \vec n\cdot\vec l=0$$

$$\sum_j W_0(\vec n_j) = \sum_j W_0\left[(-1)^j\vec n(x)b+\frac aS\vec l(x)\right] = \sum_j(-1)^j W_0\left[\vec n(x)b+(-1)^j\frac{a\vec l(x)}{S}\right] \approx \sum_j (-1)^j W_0[\vec n(x)] + \sum_j\int d\tau\, \frac{a\vec l}{S}\cdot\frac{\delta W_0(\vec n(x))}{\delta\vec n(\tau)}$$



$b\approx 1-\frac{a^2l^2}{2S^2}$

利用: $\boxed{\delta W_0(\vec n) = \frac{1}{4\pi}\int_0^\beta d\tau\; \delta\vec n\cdot(\dot{\vec n}\times\vec n)}$ $\longrightarrow$ $\frac{\delta W_0(\vec n(\tau))}{\delta\vec n(\tau)} = \frac{1}{4\pi}\left(\frac{\partial\vec n(\tau)}{\partial\tau}\times\vec n\right)$

$$W_0(\vec n(x+a))-W_0(\vec n(x)) = \int_0^\beta\left(\frac{\partial\vec n}{\partial x}a\right)\cdot\frac{\delta W(\vec n)}{\delta\vec n(\tau)}d\tau = \frac{1}{4\pi}\int_0^\beta d\tau \left(a\frac{\partial\vec n}{\partial x}\right)\cdot\left(\frac{\partial\vec n}{\partial\tau}\times\vec n\right)$$

有: $$\sum_j W_0(\vec n_j) = \frac{1}{8\pi}\int dx\int d\tau\; \partial_x\vec n\cdot(\partial_\tau\vec n\times\vec n) - \frac{1}{4\pi}\int dx\int d\tau\; \frac{\vec l}{S}\cdot(\vec n\times\partial_\tau\vec n)$$

$$\therefore\ S_B = \frac{iS}{2}\int d\tau dx\;\partial_x\vec n\cdot(\partial_\tau\vec n\times\vec n) - i\int d\tau dx\;\vec l\cdot(\vec n\times\partial_\tau\vec n)$$

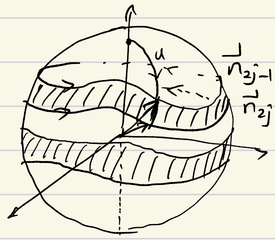
定义 $\theta=2\pi S$, $\boxed{Q=\frac{1}{4\pi}\int d\tau dx\;\vec n\cdot(\partial_x\vec n\times\partial_\tau\vec n)}$ $\longrightarrow$ 整数.

$$\boxed{S_B = i\theta Q - i\int d\tau dx\;\vec l\cdot(\vec n\times\partial_\tau\vec n)}$$

最后,

$$\boxed{S(\tau) = \underbrace{i\theta Q}_{\theta\text{-term}} \underbrace{- i\int d\tau dx\; \vec l\cdot(\vec n\times\partial_\tau\vec n)}_{-ip\dot x} + \underbrace{\frac12\int d\tau dx\left[\rho_s(\partial_x\vec n)^2+\frac{l^2}{\chi_\perp}\right]}_{H(x,p)}}$$

- θ-term 的直观说明：

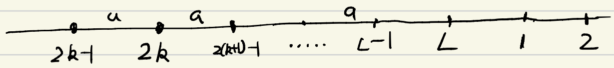


$$S_B' = 4\pi i S \sum_j (-1)^j W_0[\vec{n}(x)] = iS\sum_{k=1}^{L/2}\left[W_0(\vec{n}_{2k}(\tau)) - W_0(\vec{n}_{2k-1}(\tau))\right]4\pi$$

由于

$$4\pi W_0(\vec{n}_j) = \int_0^\beta d\tau \int_0^1 du\ \vec{n}_j(u,\tau)\cdot \frac{\partial \vec{n}_j(u,\tau)}{\partial u} \times \frac{\partial \vec{n}_j(u,\tau)}{\partial \tau} = \Omega$$

S_B' 是对图中"带"的求和



$$S_B' = \frac{iS}{2}\cdot 2a \sum_{k=1}^{L/2}\int_0^\beta d\tau\ \vec{n}\cdot\left(\frac{\Delta \vec{n}_{2j}(\tau)}{a} \times \frac{\partial \vec{n}_{2j}(\tau)}{\partial \tau}\right),\qquad \Delta n_{2k}(\tau) = \vec{n}_{2k}(\tau) - \vec{n}_{2k-1}(\tau)$$

$$= i2\pi S\cdot \frac{1}{4\pi}\int_0^L dx \int_0^\beta d\tau\ \vec{n}\cdot\left(\frac{\partial \vec{n}}{\partial x} \times \frac{\partial \vec{n}}{\partial \tau}\right)$$

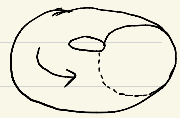
θ term，对于 $x\to x_1, \tau\to\tau_2$ 就是 skyrmion

我们也可以选择：

$$S_B' = -iS\sum_{k=0}^{\frac{L}{2}-1}\left[W_0(\vec{n}_{2k+1}(\tau)) - W_0(\vec{n}_{2k}(\tau))\right]4\pi = i2\pi S\cdot\frac{1}{4\pi}\int_0^L dx\int_0^\beta d\tau\ \vec{n}\cdot\left(\frac{\partial \vec{n}}{\partial \tau}\times\frac{\partial \vec{n}}{\partial x}\right)$$

哪个对？ 都对！

$\vec{n}_0 = \vec{n}_L$



对于光滑路径 $\vec{n}(x,\tau)$，周期边界

$\because \int dx \int d\tau\ \vec{n}\cdot\left(\frac{\partial \vec{n}}{\partial x}\times\frac{\partial \vec{n}}{\partial \tau}\right) = 4\pi Q$， Q是整数：为 $\vec{n}$ 覆盖球面的次数！ winding number

$$\therefore\ e^{-S_B'} = e^{\pm iS2\pi Q} = (-1)^{2SQ}$$

- S 整数. $e^{-S_B'} = 1$，classical，Haldane conjecture： Δ，类似于 2D O(3) NLSM without θ-term 与2D 经典模型相似. 基态顺磁，有激发态Δ.

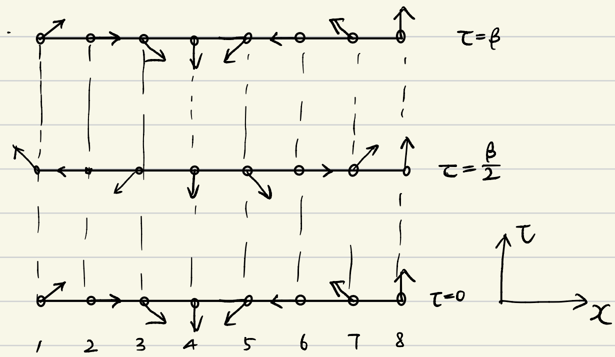
此时，拓扑θ项不重要？ 基于光滑 $\vec{n}$ 位型，是这样，但对 free boundary conditions 就不同了！

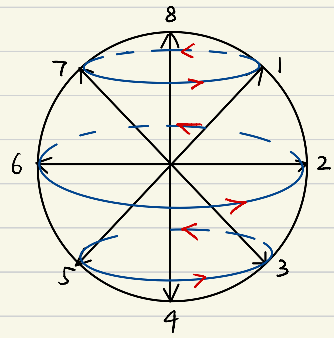
$\Rightarrow$ gapless boundary spin-$\frac{1}{2}$ excitation. see Abanov.

- S 半整数. $e^{-S_B'} = (-1)^Q$，almost Néel，类似于 $S=\frac{1}{2}$ chain，Bethe ansatz 代数decay的corr

1930年代就有此解，但是人们并不理解 $S=\frac{1}{2}$ 与 $S=1$ 的区别. ②

例.



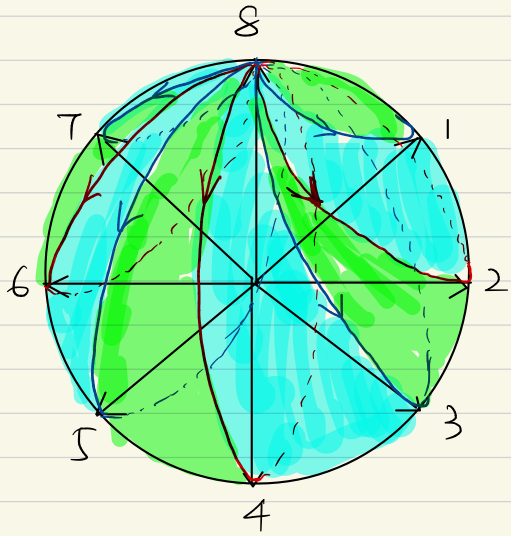


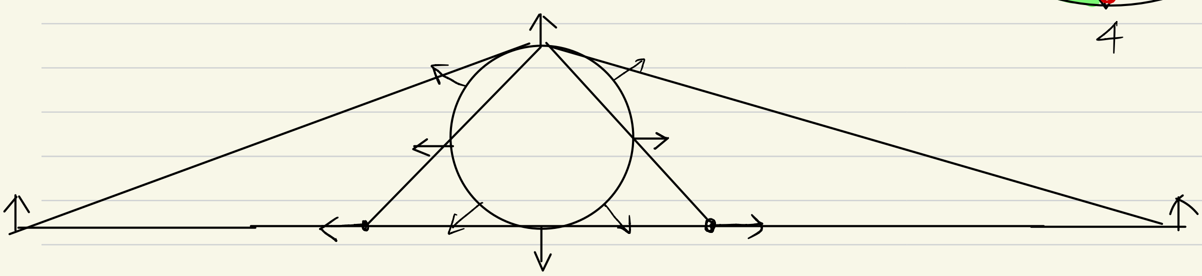
红色箭头表示自旋转动方向.

$$S_B' = \frac{iS}{2} \times 2 \times \left[\Omega(\vec{n}_2(\tau)) - \Omega(\vec{n}_1(\tau)) + \Omega(\vec{n}_4(\tau)) - \Omega(\vec{n}_3(\tau)) + \cdots + \Omega(\vec{n}_8(\tau)) - \Omega(\vec{n}_7(\tau))\right]$$

$$= \frac{iS}{2}\cdot 8\pi \quad = 2\pi i S \times 2 \qquad \int dx \int d\tau \; \vec{n}\cdot\left(\frac{\partial\vec{n}}{\partial x}\times\frac{\partial\vec{n}}{\partial \tau}\right) = 4\pi Q,$$

写成 $S_B' = i\theta Q$, 则 $Q = 2$





$$S_B' = \frac{iS}{2}\times 2\left[\Omega(\vec{n}_2(\tau)) - \Omega(\vec{n}_1(\tau)) + \Omega(\vec{n}_4(\tau)) - \Omega(\vec{n}_3(\tau)) + \cdots + \Omega(\vec{n}_8(\tau)) - \Omega(\vec{n}_7(\tau))\right]$$

$$= \frac{iS}{2}\cdot 4\pi \qquad \int dx \int d\tau \; \vec{n}\cdot\left(\frac{\partial\vec{n}}{\partial x}\times\frac{\partial\vec{n}}{\partial \tau}\right) = 4\pi Q,$$

$\Rightarrow \quad Q = 1 \qquad$ 标准 skyrmion

致谢吴新天老师，给出此图像

· 推广到 $d\geq 2$.

1D: $$\sum_j W_0(\vec{n}_j) \approx \sum_j (-1)^j W_0[\vec{n}(x)] - \frac{1}{4\pi}\int dx \int d\tau \frac{\vec{l}}{S}\cdot(\vec{n}\times\partial_\tau\vec{n})$$

其中
$$W_0(\vec{n}(x+a)) - W_0(\vec{n}(x)) = \frac{1}{4\pi}\int_0^\beta d\tau\, a\frac{\partial\vec{n}}{\partial x}\cdot\left(\frac{\partial\vec{n}}{\partial\tau}\times\vec{n}\right)$$

d维:
$$\sum_j (-1)^j W_0[\vec{n}(\vec{x})] = \sum_{j_y, j_z}(-1)^{j_y+j_z}\frac{1}{4\pi}\int\frac{dx}{2a}\int d\tau\, a\frac{\partial\vec{n}}{\partial x}\cdot\left(\frac{\partial\vec{n}}{\partial\tau}\times\vec{n}\right)$$

最终
$$\sum_j W_0(\vec{n}_j) = \sum_{j_y, j_z}(-1)^{j_y+j_z}\frac{1}{8\pi}\int dx\int d\tau\, \vec{n}\cdot\left(\frac{\partial\vec{n}}{\partial x}\times\frac{\partial\vec{n}}{\partial\tau}\right) - \frac{1}{4\pi}\int\frac{d^dx}{a^d}\int d\tau\frac{a\vec{l}}{S}\cdot(\vec{n}\times\partial_\tau\vec{n})$$

$$S_B = 4\pi i S\sum_j W_0 = i\theta Q\,\delta_{d,1} - iS a^{1-d}\int d\tau\, d^dx\, \vec{l}\cdot(\vec{n}\times\partial_\tau\vec{n})$$

$$S' = S_B + \int d\tau\, H$$

- θ- term 只有一维有作用