第十三讲

§ spin wave theory for quantum rotor model

前面我们研究了反铁磁自旋系统，推导了描述这一系统低能物理的 quantum rotor model.

现在我们在有序态附近作展开，再讨论"有序失稳"，最后考虑重整化理论来处理有序无序相变，特别是"临界点"的性质.

spin chain $\longrightarrow Z = \int \mathcal{D}\vec{n}\, \mathcal{D}\vec{l} \prod_{x,\tau} \delta(\vec{n}\cdot\vec{l})\, e^{-S}$,

$$S = i\theta Q + \int dx\, d\tau \left(-i\vec{l}\cdot(\vec{n}\times\partial_\tau\vec{n}) + H\right), \qquad H = \frac{\rho_s}{2}\left(\frac{\partial n}{\partial x}\right)^2 + \frac{1}{2\chi_\perp} l^2$$

对应的 Quantum rotor model 的 Hamiltonian 为: $(\chi_\perp \to I,\ \rho_s a \to J_e$, 离散化$)$

$$\hat{H}_{rotor} = \sum_j \frac{L_j^2}{2I} - J_e \sum_{\langle jk\rangle} \hat{n}_j \cdot \hat{n}_k$$

其中 I 为转动惯量, J_e 为最近邻作用 $(J_e > 0)$

$[n_\alpha(i), L_\beta(j)] = i\delta_{ij}\, \epsilon_{\alpha\beta\gamma}\, n_\gamma(i)$ 　　坐标与角动量对易关系

$[L_\alpha(i), L_\beta(j)] = i\delta_{ij}\, \epsilon_{\alpha\beta\gamma}\, L_\gamma(i)$ 　　角动量对易关系.

- 当 $I \to \infty$, 动能项 $\to 0$, 模型成为经典的最近邻相互作用单位矢量模型

  有序态: 所有 $\vec{n}_i$ 自发指向同一方向.

- "I" finite: 每个 rotor 的动能使它们倾向于 $l=0$ 的态. 即 $\psi(\theta,\varphi) = Y_{00} = \frac{1}{\sqrt{4\pi}}$,

  $\vec{n}_i$ 等概率指向任意方向.

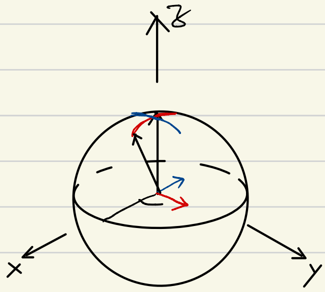
  这一趋势与相互作用的趋同竞争: 动能与势能$(\vec{n}_i\cdot\vec{n}_j)$竞争.

本节我们讨论有序态在这种竞争下的稳定性

我们采用 spin wave 理论来说明.

首先假设：所有 rotors 都指向同一方向，不失一般性，都指向 z 轴方向.

对 z 轴的偏离是小量



由于 $\vec{L_i} = \vec{n_i} \times \partial_t \vec{n_i}$ $\quad\therefore L_i$ 在 xy 面内（基本上）

n_x, n_y, L_x, L_y 是一阶小量，L_z 是二阶小量.

保留到一阶小量，我们有以下运动方程：

$$\frac{dn_x(j)}{dt} = \frac{i}{2I}[L^2(j), n_x(j)] = \frac{iL_y}{2I}[L_y, n_x] + \frac{i[L_y, n_x]}{2I}L_y + (y \to z)$$

$$= \frac{i}{2I}(-iL_y n_z - i n_z L_y + iL_z n_y + i n_y L_z) \quad \leftarrow L_z \text{ 二阶小量}$$

$$\approx \frac{1}{2I}(n_z L_y + L_y n_z) \approx \frac{L_y(j)}{I} \qquad ①$$

$$\frac{dn_y(j)}{dt} = -\frac{L_x(j)}{I} \qquad ②$$

$$\frac{dn_z(j)}{dt} = 0$$

类似地：

$$\frac{dL_x(j)}{dt} = -iJ_e \sum_{k\in j}[L_x(j), \hat{n}_j \cdot \hat{n}_k] = -iJ_e \sum_{k\in j}\{[L_x, n_y]\cdot n_y(k) + [L_x, n_z] n_z(k)\}$$

$$= J_e \sum_{k\in j} n_y(k) - J_e \sum_{k\in j} n_y(j) \qquad ③$$

$$\frac{dL_y(j)}{dt} = +J_e \sum_{k\in j} n_x(k) - J_e \sum_{k\in j} n_x(j) \qquad ④$$

$k\in j$ 表示 j 的最近邻

引入：$\phi_x(j)$ 与 $\phi_y(j)$,

$n_x(j) = \phi_x(j)$, $\quad n_y(j) = \phi_y(j)$, $\quad n_z(j) \approx 1$

$-L_x(j)$ 与 $\phi_y(j)$, $L_y(j)$ 与 $\phi_x(j)$ 正则共轭. (canonically conjugate)

因此记 $\pi_x(j) = L_y(j)$, $\quad \pi_y(j) = -L_x(j)$

满足：$\boxed{[\phi_\alpha(i), \pi_\beta(j)] = i\delta_{ij}\delta_{\alpha\beta}}$

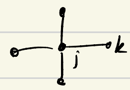
假设:

$$H_{rotor}\,(\text{linearized}) = \frac{1}{2I}\sum_j\left[\Pi_x^2(j)+\Pi_y^2(j)\right]+\frac{J_e}{2}\sum_{\langle jk\rangle}\left[\vec\phi(j)-\vec\phi(k)\right]^2$$

$$\phi^2(j)+\phi^2(k)-2\vec\phi(j)\cdot\vec\phi(k)$$

我们可以回到: 上页的 ①-④

$$\begin{cases}\dfrac{d\phi_x}{dt}=\dfrac{\Pi_x}{I},\\[2mm] \dfrac{d\phi_y}{dt}=\dfrac{\Pi_y}{I},\\[2mm] \dfrac{d\Pi_y(j)}{dt}=-J_e\sum\limits_{k\in j}\left[\phi_y(j)-\phi_y(k)\right]\\[2mm] \dfrac{d\Pi_x(j)}{dt}=-J_e\sum\limits_{k\in j}\left[\phi_x(j)-\phi_x(k)\right]\end{cases}$$



容易得到 波动方程:

$$\frac{d^2\phi_y(j)}{dt^2}=-\frac{J_e}{I}\left(2d\,\phi_y(j)-\sum_{k\in j}\phi_y(k)\right)$$

$$\frac{d^2\phi_x(j)}{dt^2}=-\frac{J_e}{I}\left(2d\,\phi_x(j)-\sum_{k\in j}\phi_x(k)\right)$$

引入 Fourier 变换:

$$\Pi_\alpha(\vec q)=\sum_j e^{i\vec q\cdot\vec x_j}\,\Pi_\alpha(j),$$

$$\phi_\alpha(\vec q)=\sum_j e^{-i\vec q\cdot\vec x_j}\,\phi_\alpha(j)$$

我们来到 q 空间. 易证

$$[\phi_\alpha(\vec q_1),\Pi_\beta(\vec q_2)]=\sum_{jk}e^{-i\vec q_1\cdot\vec x_k}e^{i\vec q_2\cdot\vec x_j}[\phi_\alpha(k),\Pi_\beta(j)]$$

$$\boxed{[\phi_\alpha(\vec q_1),\Pi_\beta(\vec q_2)]=i\delta_{\alpha\beta}N_s\,\delta_{\vec q_1,\vec q_2}}$$

$$=i\delta_{\alpha\beta}\,\delta^d(\vec q_1-\vec q_2)\quad \text{当 } N_s\to\infty$$

这样,

$$H_{rotor}\,(\text{linearized})=\frac{1}{N_s}\sum_q\frac{1}{2I}\left(|\Pi_x(\vec q)|^2+|\Pi_y(\vec q)|^2\right)+\frac{1}{N_s}\sum_q\frac{J_e}{2}b^2(\vec q)\left(|\phi_x(\vec q)|^2+|\phi_y(\vec q)|^2\right)$$

其中 $\boxed{b^2(\vec q)=\left(2d-2\cos(q_xa)-2\cos(q_ya)-\cdots\right)}$

证:

$$\frac{1}{N_s}\sum_q|\Pi_\alpha(\vec q)|^2=\frac{1}{N_s}\sum_{j}\sum_{k}\sum_{q}e^{i\vec q\cdot x_j-iq x_k}\,\Pi_\alpha(j)\,\Pi_\alpha(k)=\sum_j\Pi_\alpha^2(j)$$

$$\sum_j\phi^2(j)=\frac{1}{N_s}\sum_q|\phi(q)|^2=\sum_j\phi^2(j+1)=\sum_j\phi^2(j-1)$$

$$\sum_j \phi(j)[\phi(j+1)+\phi(j-1)] = \frac{1}{N_s^2}\sum_j \Big\{\sum_q e^{iqx_j}\phi(q)\Big[\sum_{q'} e^{iq'(x_j+1)}\phi(q') + \sum_{q''} e^{iq''(x_j-1)}\phi(q'')\Big]\Big\}$$

$$= \frac{1}{N_s^2} \times N_s \sum_q |\phi(q)|^2 (e^{iq} + e^{-iq})$$

因此：

$$\boxed{H = \frac{1}{N_s}\sum_{\vec{q}}\Big[\frac{|\vec{\pi}(\vec{q})|^2}{2I} + \frac{1}{2} I \omega^2(\vec{q})|\phi(\vec{q})|^2\Big]}$$

with $\boxed{\omega^2(\vec{q}) \equiv \frac{J_e}{I} b^2(\vec{q})}$ ，每个 mode $\vec{q}$ 都是一个谐振子. （$\omega(\vec{q})$ 线性）

引入

$$\boxed{\begin{aligned} a_\alpha(\vec{q}) &= \frac{1}{\sqrt{2N_s}}\Big(\sqrt{I\omega(\vec{q})}\,\phi_\alpha(\vec{q}) + i\frac{\pi_\alpha(-\vec{q})}{\sqrt{I\omega(\vec{q})}}\Big) \\ a^\dagger_\alpha(\vec{q}) &= \frac{1}{\sqrt{2N_s}}\Big(\sqrt{I\omega(\vec{q})}\,\phi_\alpha(\vec{q}) - i\frac{\pi_\alpha(\vec{q})}{\sqrt{I\omega(\vec{q})}}\Big)\end{aligned}}$$

对比一维谐振子：$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$

定义 $Q = \sqrt{\frac{m\omega}{\hbar}}\, x$ ，$P = \frac{p}{\sqrt{m\hbar\omega}}$. $[Q, P] = i$

$\hat{a} = \frac{1}{\sqrt{2}}(Q + iP)$, $\hat{a}^+ = \frac{1}{\sqrt{2}}(Q - iP)$

$[a, a^+] = 1$, $H = \hbar\omega(a^+a + \frac{1}{2})$

由于 $\omega(\vec{q}) = \omega(-\vec{q})$ ，有

$$\boxed{[a_\alpha(\vec{q}_1), a^\dagger_\beta(\vec{q}_2)] = \delta_{\vec{q}_1\vec{q}_2}\delta_{\alpha\beta}}$$

$$\boxed{H = \sum_{\vec{q}\alpha} \big(a^\dagger_\alpha(\vec{q})\, a_\alpha(\vec{q}) + \tfrac{1}{2}\big)\, \omega(\vec{q})}$$

$$a^\dagger_\alpha(q)\, a_\alpha(\vec{q}) = \frac{1}{2}(\phi(-q) - i\pi(q))(\phi(q) + i\pi(-q))$$
$$= \frac{1}{2}|\phi|^2 + \frac{1}{2}|\pi|^2 + [i\phi(-q)\pi(-q) - i\pi(q)\phi(q)]\frac{1}{2}$$
$$+ a^\dagger_\alpha(-q)\, a_\alpha(-q)$$
$$= \frac{1}{2}|\phi|^2 + \frac{1}{2}|\pi|^2 + [i\phi(q)\pi(q) - i\pi(-q)\phi(-q)]\frac{1}{2}$$

因此 对给定的 $\vec{q}$ 与 $\alpha = x, y$. (mode) 每一对 $(a_\alpha(\vec{q}), a^\dagger_\alpha(\vec{q}))$ 给出一组独立的产生、消灭算符，

对应的 粒子数 算符：

$$\boxed{\hat{n}_\alpha(\vec{q}) = a^\dagger_\alpha(\vec{q})\, a_\alpha(\vec{q})}$$

计算 处于 $(\vec{q}, \alpha)$ mode 的 自旋波 粒子 (magnons) 的数目.

于是：

$$\boxed{H = \sum_{\vec{q},\alpha} \omega(\vec{q})\big(\hat{n}_\alpha(\vec{q}) + \tfrac{1}{2}\big)}$$

GS: magnons 的 真空： $n_\alpha(\vec{q}) = 0$. 对应 所有 $\hat{n}(j)$ 指向 $\hat{z}$.

非常类似 phonons in solid，或 photons in microwave cavity

外部扰动可以 create these magnons

玻色统计：每个模式是一个微观状态