第十五讲

前面研究了反铁磁 Heisenberg chain.

横场 Ising chain (Transverse field Ising chain)

$$\boxed{\hat H = -J\sum_i \hat\sigma_i^z \hat\sigma_{i+1}^z - h\sum_i \hat\sigma_i^x}$$

- $[\hat\sigma_i^\alpha, \hat\sigma_j^\beta] = \delta_{ij}\, 2i\, \varepsilon^{\alpha\beta\gamma}\hat\sigma_i^\gamma$ ← $[\sigma_x, \sigma_y] = 2i\sigma_z$

- Z_2 对称性: $\hat U = \prod_i e^{i\pi \frac{\sigma_i^x}{2}} = \prod_i (i\sigma_i^x)$

由于 $\hat U \sigma_i^z \hat U^+ = -\sigma_i^z$ $\left(\because\ i\sigma_i^x \sigma_i^z (-i\sigma_i^x) = -(\sigma_i^x)^2\sigma_i^z\right)$

$\hat U \sigma_i^x U^+ = \sigma_i^x$

$\therefore\ UHU^+ = H \Rightarrow$ 若 $H|E\rangle = E|E\rangle$, 则 $H\,U|E\rangle = UHU^+U|E\rangle = E\,U|E\rangle$

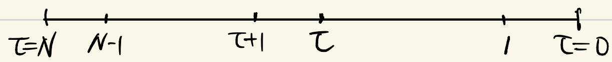
- 当 $h=0$, $|GS\rangle = |\uparrow\uparrow\cdots\uparrow\rangle = \prod_{i=1}^{N_s}|\sigma_i=1\rangle$ or $|GS\rangle = |\downarrow\downarrow\cdots\downarrow\rangle = \prod_{i=1}^{N_s}|\sigma_i=-1\rangle$

当 $J=0$ $|GS\rangle = |\rightarrow\cdots\cdots\rightarrow\rightarrow\rangle = \prod_{i=1}^{N_s}\frac{1}{\sqrt2}(|+\rangle+|-\rangle)_i$

路径积分:

$$Z = \mathrm{Tr}\,(e^{-\beta\hat H})$$

$$= \sum_{\{\sigma\}} \langle \underbrace{\sigma_1\cdots\sigma_{N_s}}_{\{\sigma\}}|\, e^{-\epsilon H} \cdots\cdots e^{-\epsilon H}\, |\underbrace{\sigma_1\cdots\sigma_{N_s}}_{\{\sigma\}}\rangle \qquad \epsilon = \beta/N$$



插入完备性关系: $\sum_{\{\sigma\}} |\{\sigma\}\rangle\langle\{\sigma\}| = \mathbb{1}$ 例: 两自旋, $(|\uparrow\rangle_1\langle\uparrow| + |\downarrow\rangle_1\langle\downarrow|)\otimes(|\uparrow\rangle_2\langle\uparrow| + |\downarrow\rangle_2\langle\downarrow|)$

$= |\uparrow\uparrow\rangle\langle\uparrow\uparrow| + |\downarrow\downarrow\rangle\langle\downarrow\downarrow| + |\uparrow\downarrow\rangle\langle\uparrow\downarrow| + |\downarrow\uparrow\rangle\langle\downarrow\uparrow| = \mathbb{1}$

$$Z = \sum_{\{\sigma\}_0,\{\sigma\}_1}\cdots\sum_{\{\sigma\}_{N-1}} \langle\{\sigma\}(1)|e^{-\epsilon H}|\{\sigma\}(N)\rangle\langle\{\sigma\}(N)|e^{-\epsilon H}\cdots\cdots\langle\{\sigma\}(1)|e^{-\epsilon H}|\{\sigma\}(0)\rangle$$

关键是计算 $\boxed{\langle\{\sigma\}(\tau+1)|e^{-\epsilon H}|\{\sigma\}(\tau)\rangle = \langle\{\sigma\}(\tau+1)|e^{\epsilon J\sum_i \sigma_i^z\sigma_{i+1}^z + \epsilon h\sum_i \sigma_i^x}|\{\sigma\}(\tau)\rangle}$ ①

困难: $\sigma_i^z\sigma_{i+1}^z$ 与 σ_i^x 不对易, 但: $e^{a(A+B)} = e^{aA}e^{aB}e^{a^2[A,B]}$

当 $a\to0$, $e^{a(A+B)} = e^{aA}\cdot e^{aB} + O(a^2)$ ← Trotter 分解

$\therefore$ ① $= \langle\{\sigma\}(\tau+1)|\, e^{\epsilon J\sum_i\sigma_i^z\sigma_{i+1}^z}\, e^{\epsilon h\sum_i\sigma_i^x}\,|\{\sigma\}(\tau)\rangle$

注意到: $\langle\sigma(\tau+1)|\, e^{\epsilon h\hat{\sigma}_x}\, |\sigma(\tau)\rangle = \langle\sigma(\tau+1)|\, (ch(\epsilon h)\cdot \mathbb{1} + sh(\epsilon h)\,\hat{\sigma}_x)\, |\sigma(\tau)\rangle$

$$= ch(\epsilon h)\,\delta_{\sigma_\tau\sigma_{\tau+1}} + sh(\epsilon h)\,(1-\delta_{\sigma_\tau\sigma_{\tau+1}})$$

$$= A\, e^{J_\tau(\sigma_\tau\sigma_{\tau+1}-1)}$$

其中 $e^{-2J_\tau} = th(\epsilon h)$, $A = ch(\epsilon h)$

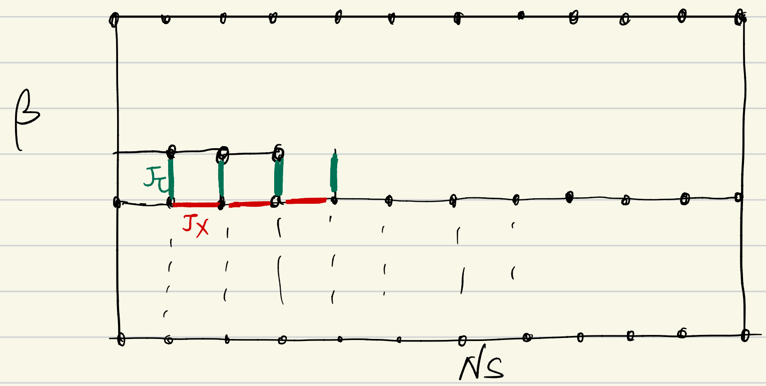
因此 $① = e^{J_x\sum_i \sigma^i_{\tau+1}\sigma^{i+1}_{\tau+1}}\; e^{J_\tau\sum_i \sigma^i_{\tau+1}\sigma^i_\tau}\cdot A$

其中 $\boxed{J_x \equiv \epsilon J,\quad e^{-2J_\tau} = th(\epsilon h),\quad A\ \text{常数}.}$ $\epsilon = \beta/N$

最后, $\epsilon\to 0$ 时. $J_\tau = -\frac{1}{2}\log(\epsilon h)$

$$Z = \sum_{\{\sigma\}} e^{-H_c(\{\sigma\})}$$

$\boxed{H_c = -J_x\sum_{i\tau}\sigma^i_\tau\sigma^{i+1}_\tau - J_\tau\sum_{i\tau}\sigma^i_{\tau+1}\sigma^i_\tau}$, $\sigma^i_\tau = \pm 1$. # of spins $= N\times N_s$



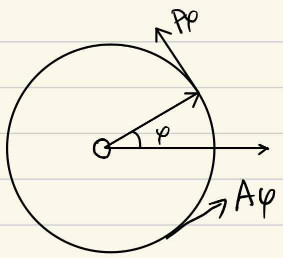
发现 1D TFIM = 2D 经典 Ising model, but 各向异性, ∵ J_x 一般不等于 J_τ

- $D \to D+1$ 的 量子-经典 mapping.
- 量子 model 的 Z_2 symmetry 体现为 $\{\sigma\}(\tau) \to -\{\sigma\}(\tau)$, H_c不变.

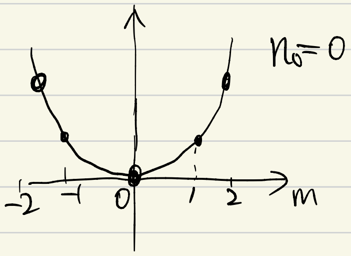
- $S=\frac{1}{2}$ XY chain $\quad H = J\sum_i \left(\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y\right)$

U(1) 对称性: $\hat U = \prod_i e^{i\theta \frac{\sigma_i^z}{2}}$, $\hat U^\dagger H \hat U = H$, $+ Z_2$

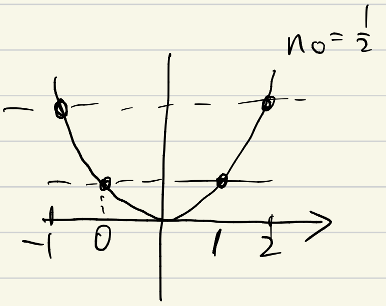
§ 带电粒子在圆环上.



- 位置 $\hat\varphi$, 动量 $\hat P_\varphi$, $[\hat\varphi, \hat P_\varphi] = i\hbar$.
- 考虑中心有磁通量穿过, 因此有 A_φ (矢势), $2\pi R A_\varphi = \Phi$
- $H = \frac{1}{2M}\left(P_\varphi - \frac{q}{c}A_\varphi\right)^2$, $\quad P_\varphi = -i\hbar\frac{\partial}{R\partial\varphi}$, $\quad L_z = -i\hbar\frac{\partial}{\partial\varphi}$
- $A_\varphi=0$ 时. H_0 本征态 $\psi_m(\varphi) = \frac{e^{im\varphi}}{\sqrt{2\pi}}$, $\quad m=0,\pm1,\pm2.$ 设R=1



$$\boxed{E_m = \frac{m^2}{2M}}$$

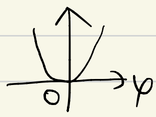


- 当 $A_\varphi\neq0$ 时, 简写 $H=\frac{1}{2M}(P_\varphi-n_0)^2$, $\quad n_0=\frac{q}{c}A_\varphi$.

H 本征态不变, $\frac{e^{im\varphi}}{\sqrt{2\pi}}$, $\quad \boxed{E_m = \frac{1}{2M}(m-n_0)^2}$

- 加入外电场: $H = \frac{1}{2M}(P_\varphi - n_0)^2 + f(\hat\varphi)$

比如 $f(\hat\varphi)=\frac{1}{2}\hat\varphi^2$



$$Z_R = \mathrm{Tr}(e^{-\beta H})$$

$$= \int \mathcal{D}\varphi(\tau)\, \langle\varphi(N)|e^{-\epsilon H}|\varphi(N-1)\rangle\langle\varphi(N-1)| \cdots\cdots \langle\varphi(1)|e^{-\epsilon H}|\varphi(0)\rangle$$

$$|\varphi(N)\rangle = |\varphi(0)\rangle$$

再插入 $\sum_m |m\rangle\langle m|$ 在每个 τ 时刻

$$Z_R = \int\mathcal D\varphi(\tau)\sum_{\{m\}} \langle\varphi(N)|m(N)\rangle\langle m(N)|e^{-\epsilon H}|\varphi(N-1)\rangle \cdots \langle\varphi_1|m_0\rangle\langle m_0|e^{-\epsilon H}|\varphi(0)\rangle$$

$$\langle\varphi(\tau+1)|m(\tau)\rangle = \frac{1}{\sqrt{2\pi}}\, e^{i m_\tau \varphi_{\tau+1}}$$

$$\boxed{\begin{aligned}\langle m(\tau)|e^{-\epsilon H}|\varphi(\tau)\rangle &= \langle m(\tau)|e^{-\frac{\epsilon}{2M}(P_\varphi-n_0)^2}\, e^{-\epsilon f(\hat\varphi)}|\varphi(\tau)\rangle \\ &\approx e^{-\frac{\epsilon}{2M}(m(\tau)-n_0)^2}\, e^{-\epsilon f(\varphi(\tau))}\, e^{-i m(\tau)\varphi(\tau)}\end{aligned}}$$

$$\therefore Z = \int\mathcal D\varphi(\tau)\sum_{\{m\}} e^{-S} \qquad S = \sum_{\tau=0}^{N-1} -i m(\tau)[\varphi(\tau+1)-\varphi(\tau)] + \epsilon f(\varphi(\tau)) + \frac{\epsilon}{2M}(m(\tau)-n_0)^2$$

先完成对 $m(\tau)$ 的求和：（在某个 τ 时刻）

$$\sum_m e^{-(-i m\Delta\varphi + \frac{\epsilon}{2M}(m-n_0)^2)} = \sum_{m=-\infty}^{\infty} e^{i\Delta\varphi m - \frac{\epsilon}{2M}(m-n_0)^2} \quad ①$$

利用 Poisson 求和：
$$\boxed{\sum_{m=-\infty}^{\infty} f(m) = \sum_{k=-\infty}^{\infty} \int dm\, e^{i2\pi k m} f(m)}$$

$$① = \sum_k \int dm\, e^{i2\pi m k}\, e^{i\Delta\varphi m}\, e^{-\frac{\epsilon}{2M}(m-n_0)^2}$$

$$= \sum_{k=-\infty}^{\infty} \int dm\, e^{-\frac{\epsilon}{2M}m^2}\, e^{i2\pi k(m+n_0)}\, e^{i\Delta\varphi(m+n_0)} \qquad (m-n_0 \to m)$$

$$= \sum_k e^{i2\pi k n_0}\, e^{i\Delta\varphi n_0} \int dm\, e^{-\frac{\epsilon}{2M}m^2 + i(2\pi k+\Delta\varphi)m}$$

$$= \sum_k e^{i2\pi k n_0}\, e^{i\Delta\varphi n_0}\, e^{-\frac{(2\pi k+\Delta\varphi)^2 2M}{4\epsilon}} \approx e^{-i\Delta\varphi n_0 - \frac{\Delta\varphi^2 M}{2\epsilon}}$$

（$k\neq 0$ 项 $\to 0$ 当 $\epsilon \to 0$）

连续极限下： 且注意到 $\varphi(\beta)-\varphi(0) = 2\pi Q$， $Q\in\mathbb{Z}$ 缠绕数 (Winding Number)

$$S = \sum_{\tau=0}^{N_\tau-1} -i\epsilon n_0 \frac{\Delta\varphi}{\epsilon} + \frac{M}{2}\epsilon\frac{\Delta\varphi^2}{\epsilon^2} + \epsilon f(\varphi)$$

$$= \int_0^\beta d\tau \left(-i n_0 \frac{d\varphi}{d\tau} + \underbrace{\frac{M}{2}\left(\frac{d\varphi}{d\tau}\right)^2 + f(\varphi)}_{L}\right)$$

第1项积分：$-i(n_0\, 2\pi)\, Q \longrightarrow$ topological θ-term

$n_0 = \frac{1}{2}$， 等价于 $S = \frac{1}{2}$.