第十七讲

《Statistical Field Theory》 David Tong

现代物理的范式(paradigm)围绕 two deep facts about the Universe:

- Nature is organised by symmetry,

Landau: phases 由(破缺的)对称性刻画

我们在讨论相变问题时发展的所有 ideas 都可直接用到粒子物理、宇宙学 and beyond!

- Nature is organised by scale

order:

$\boxed{\text{little things affect big things,}}$ 粒子→核→原子→凝聚态,化学→……

不会相反,所以太阳里没有银河系.

But, another aspect:

little things affect big things, but they rarely affect very big things, but slightly bigger things, and so on.

As you go up the chain, you lose the information about what came long before.

研究椋鸟群聚的动物学家不需要研究 Higgs boson 的动力学

牛顿, Einstein 不需要知道 Quantum gravity 就可以写下他们的方程!

$\boxed{\text{RG provides a framework that makes these ideas concrete.}}$

It describes physics at different scales.

The right way to understand both the Higgs boson and the flocking of starlings is through the language of the RG

symmetry and scale: 决定了我们对物理的思考.

都来自于一个 simple Question: 烧开水时发生了什么?

• 原胞哈密顿模型.

Ising, XY, Heisenberg: $\vec{\sigma}_c$, c 表示原胞.

空间维数 d 与自旋维数 n　　哈密顿: $H_c(\sigma)$

$$\vec{\sigma}_c = (\sigma_{1c}, \sigma_{2c}, \cdots, \sigma_{nc})$$

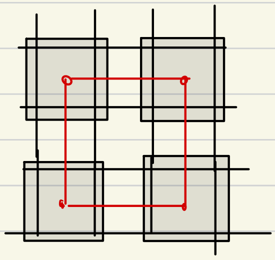
$$Z = \int e^{-\frac{H_c(\sigma)}{T}} \prod_{i,c} d\sigma_{ic}$$, nL^d 个变量, Ising 积分→求和

$$P(\sigma) = \frac{e^{-\frac{H_c(\sigma)}{T}}}{Z}$$

物理量, 比如 自旋关联:

$$\langle \sigma_{i\vec{x}} \sigma_{j\vec{y}} \rangle = \int \sigma_{i\vec{x}} \sigma_{j\vec{y}} P(\sigma) \prod_{i,c} d\sigma_{ic}$$

• 块哈密顿 和 卡丹诺夫 变换



$b=2$　　b^d 个原胞自旋的平均 → 块自旋 block spin

block 之间距离为 b

设 $P(g_1, g_2)$ 是两个随机变量 g_1, g_2 的几率分布, 定义 $g = \frac{1}{2}(g_1 + g_2)$

问 g 的几率分布是怎样的?

$$\boxed{P'(g) = \int dg_1\, dg_2\, \delta\left(g - \tfrac{1}{2}(g_1 + g_2)\right) P(g_1, g_2) = \left\langle \delta\left(g - \tfrac{1}{2}(g_1+g_2)\right)\right\rangle_P} \quad (1)$$

任意 $f(g)$ 的平均值, 比如 $\langle g^2 \rangle$ 都可以用 $P'(g)$ 计算, 也可用 $P(g_1, g_2)$ 来算!

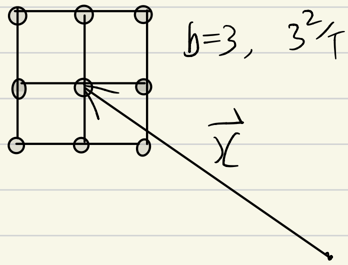
$$\langle g^2 \rangle_{P'} = \int g^2 P'(g)\, dg = \int g^2 \left[\int \delta\left(g - \tfrac{1}{2}(g_1+g_2)\right) P(g_1, g_2)\, dg_1 dg_2\right] dg$$

$$= \int \left(\tfrac{1}{2}(g_1+g_2)\right)^2 P(g_1, g_2)\, dg_1 dg_2 = \left\langle \left[\tfrac{1}{2}(g_1+g_2)\right]^2 \right\rangle_P$$

推广

$$\langle f(g) \rangle_{P'} = \int dg\, f(g)\, P'(g) = \left\langle f\left(\tfrac{g_1+g_2}{2}\right)\right\rangle_P \quad (2)$$

定义 块自旋 (block spin): $\vec{\sigma}_{\vec{x}} = \frac{1}{b^d}\sum_{c\in\vec{x}}\vec{\sigma}_c$,



$b=3$, 3^2个 cell

- $\vec{x}$ 为块中心坐标

根据(1)式, block spin 的概率分布 $P'(\vec{\sigma}_{\vec{x}})$ 为:

$$P'(\vec{\sigma}_{\vec{x}}) = \left\langle \prod_{i\vec{x}} \delta\left(\sigma_{i\vec{x}} - \frac{1}{b^d}\sum_{c\in\vec{x}}\sigma_{ic}\right)\right\rangle_{P(\vec{\sigma}_c)}$$

$$= \frac{1}{Z}\int e^{-H_c[\vec{\sigma}_c]/T}\prod_{i\vec{x}}\delta\left(\sigma_{i\vec{x}} - \frac{1}{b^d}\sum_{c\in\vec{x}}\sigma_{ic}\right)\prod_{jc}d\sigma_{jc}$$

$$= \frac{1}{Z}e^{-H_b[\vec{\sigma}_{\vec{x}}]/T} \qquad (2)$$

$j, i = 1, \cdots, n$ 分量, c 取所有原胞(格点),

$H[\vec{\sigma}_{\vec{x}}]$ 就是块哈密顿 (如果没有δ函数限制, $H[\vec{\sigma}_{\vec{x}}]$ 就是总自由能)

对块自旋位形求积分: (代入(2)式, 可证明)

$$Z = \int e^{-\frac{H_b[\vec{\sigma}_{\vec{x}}]}{T}}\prod_{i\vec{x}} d\sigma_{i\vec{x}}$$

- 配分函数不变, 随机变量变成了块自旋
- 描述 ba 尺度之上的变化: 分辨率为 ba. (a为晶格常数)
- 如果关心大于 ba 尺度的平均值, $H_c[\vec{\sigma}_c]$ 与 $H_b[\vec{\sigma}_{\vec{x}}]$ 等效.

$$\because \quad \langle f(\xi)\rangle_{P'} = \int d\xi\, f(\xi)\, P'(\xi) = \left\langle f\left(\frac{\sigma_1+\sigma_2}{2}\right)\right\rangle_P$$

下面讨论另一种定义块自旋的方法.

先考虑下面的几率公式: g_1, g_2 随机变量, $P(g_1, g_2)$ 分布

只关心 g_1, 问它的几率分布为?

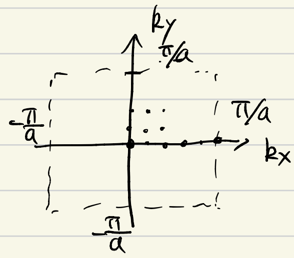
$$\boxed{P'(g_1) = \int dg_2 \, P(g_1, g_2)}$$

自旋位型可以在傅立叶空间讨论.

$$\boxed{\vec{\sigma}_{\vec{k}} = \sum_{\vec{r}} e^{-i\vec{k}\cdot\vec{r}} \vec{\sigma}_{\vec{r}}}$$

逆:

$$\boxed{\vec{\sigma}_{\vec{r}} = \frac{1}{L^d} \sum_{\vec{k}} e^{i\vec{k}\cdot\vec{r}} \vec{\sigma}_{\vec{k}}}$$



$\vec{k} = \frac{2\pi}{La}\vec{n}$, $n_x, n_y, \cdots = -\frac{L}{2}+1, \cdots, \frac{L}{2}$ 第一布里渊区. $|k_x|, |k_y| \leq \frac{\pi}{a}$

• 当 $L \to \infty$, $\vec{k}$ modes 连续. 变成对 $\vec{k}$ 积分.

$$\boxed{\vec{\sigma}_{\vec{r}} = \int \frac{d^d k}{(\frac{2\pi}{a})^d} e^{i\vec{k}\cdot\vec{r}} \vec{\sigma}_{\vec{k}}} \qquad \left(\sum_{\vec{k}} = \int \frac{d^d k}{(\frac{2\pi}{La})^d}\right)$$

$$\boxed{Z = \int e^{-H(\vec{\sigma}_{\vec{r}})/T} \prod_{i,\vec{r}} d\sigma_{i,\vec{r}} = |J| \int e^{-H(\vec{\sigma}_{\vec{k}})/T} \prod_{i,\vec{k}} d\sigma_{i,\vec{k}}} \qquad (3)$$

例: $H = y_1 \cdot y_2$, 定义 $y_1 = x_1 + x_2$, $y_2 = x_1 - x_2$, $\Rightarrow H = x_1^2 - x_2^2$

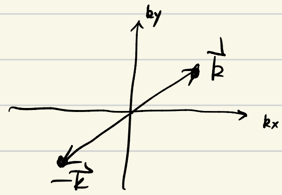
$$Z = \int e^{-y_1 \cdot y_2} dy_1 dy_2 = |J| \int e^{-(x_1^2 - x_2^2)} dx_1 dx_2$$

$x = r\cos\theta$, $y = r\sin\theta$, $dxdy = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} dr d\theta$

$$J = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2$$

• 常数J不重要: $P(\vec{\sigma}_{\vec{k}}) = \frac{1}{Z} |J| e^{-H(\vec{\sigma}_{\vec{k}})/T}$, 约去

$\sigma_{i\vec k} = (\sigma_{i-\vec k})^*$ ∵ $\sigma_{i\vec k} = \sum_{\vec c} e^{i\vec k\cdot\vec c}\,\sigma_{i\vec c}$，$\sigma_{i\vec c}$ 实



$$\boxed{d\sigma_{i\vec k}\, d\sigma_{i-\vec k} = d\,\mathrm{Re}\,\sigma_{i\vec k}\; d\,\mathrm{Im}\,\sigma_{i\vec k}}$$

给定 i： L^d 个 $\sigma_{i\vec c}$ 对应 L^d 个 $\sigma_{i\vec k}$，

一对 $\vec k$ 与 $-\vec k$ 对应一对 $\mathrm{Re}\,\sigma_{i\vec k}$ 和 $\mathrm{Im}\,\sigma_{i\vec k}$

$$\int_{-\infty}^{\infty} dx\, e^{-\frac12 ax^2} = \left(\frac{2\pi}{a}\right)^{\frac12}$$

总之：

$$\int d\sigma_{i\vec k}\, d\sigma_{i-\vec k}\, e^{-\frac a2 |\sigma_{i\vec k}|^2} = \int d(\mathrm{Re}\,\sigma_{i\vec k})\, d(\mathrm{Im}\,\sigma_{i\vec k})\, e^{-\frac a2[(\mathrm{Re}\,\sigma_{i\vec k})^2+(\mathrm{Im}\,\sigma_{i\vec k})^2]} = \left(\frac{2\pi}{a}\right)^{\frac12}\cdot\left(\frac{2\pi}{a}\right)^{\frac12}$$

每个 $\vec k$ 贡献一个因子 $\left(\frac{2\pi}{a}\right)^{\frac12}$.

如果我只对 $|k|<\Lambda$（某个小于 $\frac{\pi}{a}$ 的截断）的分量 σ_{ik} 感兴趣，则

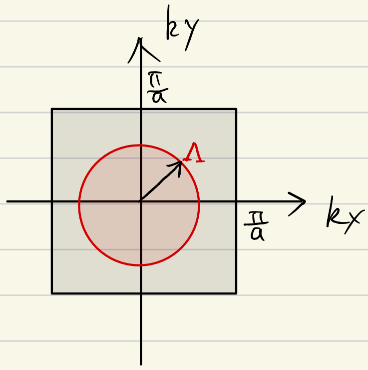
$$\boxed{P'(\vec\sigma_{\vec k}) = \frac1Z\int e^{-\hat H[\vec\sigma]/T}\prod_{i,|k|>\Lambda} d\sigma_{i\vec k} \equiv \frac{e^{-H[\vec\sigma_{|k|<\Lambda}]/T}}{Z}} \qquad (4)$$

• P' 描述半径为 Λ 的 $\vec k$ 球内的 $\sigma_{i\vec k}$ 分布，$H[\vec\sigma_{|k|<\Lambda}]$ 是相应的块哈密顿

Λ 给出了一个有效的 $a'=ba$：$\Lambda=\frac{\pi}{a'}$，或 $ba=\frac{\pi}{\Lambda}$

在小于 a' 的尺度 (block) 下，没有自旋取值.（block 由 b^d 个原胞构成）

与前面平均 block 内自旋平均效果相当.



这里块自旋

$$\boxed{\vec\sigma(\vec x) = \frac{1}{L^d}\sum_{|k|<\Lambda}\vec\sigma_k\, e^{i\vec k\cdot\vec x}}$$

(2) 与 (4) 给出 Cell Hamiltonian 到 block Hamiltonian 的程序: Kadanoff 变换

$$\frac{H_b[\sigma]}{T} = \hat{K}_b \cdot \frac{H_c[\sigma]}{T}$$

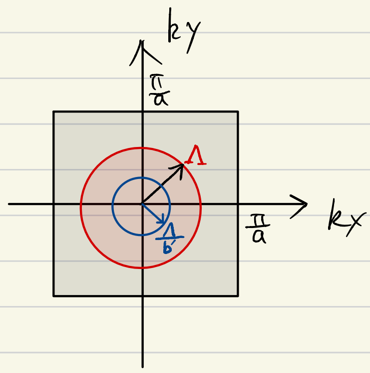
$\hat{K}_b$ 表示块为 b 个原胞的 Kadanoff 变换， $K_1 = \mathbb{1}$

在 $H_b[\sigma]$ 基础上，再做 $K_{b'}$: 把 b' 个块自旋平均成"大"块自旋:

$$\sigma_{i\vec{x}'} = \frac{1}{b'^d} \sum_{\vec{x}\in\vec{x}'} \sigma_{i\vec{x}}$$

$\vec{x}'$ 标志"大" block，间隔是 $b' \cdot ba$

$$\frac{H_{b'}[\sigma]}{T} = \hat{K}_{b'} \cdot \frac{H_b[\sigma]}{T}$$



也可以采取从 Λ 区域中积掉 $|k| > \frac{\Lambda}{b'}$ 的方法:

$$e^{-\frac{H_{b'}[\sigma]}{T}} = \int e^{-\frac{H_b[\sigma]}{T}} \prod_{i,\,|k|>\Lambda/b'} d\sigma_{i,\vec{k}}$$

这样定义的分辨率为 $b' \cdot ba$

$$\frac{H_{b'}[\sigma]}{T} = \hat{K}_{b'} \cdot \frac{H_b[\sigma]}{T}$$

显然: $\hat{K}_{b'} \cdot \frac{H_b(\sigma)}{T} = \hat{K}_{b'} \cdot \left(\hat{K}_b \cdot \frac{H_c(\sigma)}{T}\right) = K_{b'b} \cdot \frac{H_c(\sigma)}{T}$

$\hat{K}_{b'b}$ 是用 $b'b$ 个原胞构成 block.

$$\therefore \boxed{\hat{K}_{b'} \hat{K}_b = \hat{K}_{b'b}}$$

§ Landau-Ginzburg 哈密顿 或作用量

Ginzburg 和 Landau 假定了一个块哈密顿量形式

$$\boxed{\frac{H[\sigma]}{T} = \int d^d x \left[a_0 + \frac{1}{2} a_2 \sigma^2 + \frac{1}{4} a_4 \sigma^4 + \frac{1}{2} c (\nabla \sigma)^2 + \cdots\right]} \qquad (1)$$

$$\sigma^2 \equiv \vec{\sigma}(\vec{x}) \cdot \vec{\sigma}(\vec{x}) = \sum_{i=1}^{n} (\sigma_i(\vec{x}))^2$$

$$\sigma^4 \equiv (\sigma^2)^2$$

$$(\nabla\sigma)^2 = \sum_{i=1}^{n} \sum_{\alpha=1}^{d} \left(\frac{\partial \sigma_i}{\partial x_\alpha}\right)^2 \qquad \left(或 \sum_i [\nabla\sigma_i \cdot \nabla\sigma_i]\right)$$

- $a_0(T)$, $a_2(T)$, $a_4(T)$, $c(T)$ 都是温度 T 的函数

(1) 式可理解为：

$$\frac{H[\sigma]}{T} = (ba)^d \sum_{第\vec{x}个 block} \left[a_0 + \frac{a_2}{2} \sigma_{\vec{x}}^2 + \frac{a_4}{4} \sigma_{\vec{x}}^4 + \frac{c}{4} \sum_{\vec{y}}{}' \frac{(\vec{\sigma}_{\vec{x}} - \vec{\sigma}_{\vec{x}+\vec{y}})^2}{(ba)^2}\right]$$

$\vec{y}$ 是 $\vec{x}$ 的 $2d$ 个近邻

(1) 式也可以在 $\vec{k}$ 空间写出

$$\frac{H[\sigma]}{T} = a_0 L^d + a_2 \int d^d x \sum_{|\vec{k}|<\Lambda} e^{i\vec{k}\cdot\vec{x}} \sigma_{i\vec{k}}$$

$$\int d^d x\, \sigma^2 = \int d^d x \sum_i \sigma_i(\vec{x}) \cdot \sigma_i(\vec{x}) = \frac{1}{L^{2d}} \sum_i \int d^d x \left(\sum_{|\vec{k}|<\Lambda} e^{i\vec{k}\cdot\vec{x}} \sigma_{i\vec{k}}\right) \left(\sum_{|\vec{k}'|<\Lambda} e^{i\vec{k}'\cdot\vec{x}} \sigma_{i\vec{k}'}\right)$$

$$= \frac{1}{L^d} \sum_i \sum_{k<\Lambda} \sigma_{i\vec{k}} \cdot \sigma_{i-\vec{k}} \qquad \left(\int dx\, e^{ikx} \to \int \frac{dx}{(ba)} (ba) e^{ikx} = ba \sum_{block} e^{ikx} = ba\left(\frac{L}{ba}\right)\delta_{k,0}\right)$$

$$\int d^d x\, c(\nabla\sigma)^2 = \frac{1}{L^d} \sum_i \sum_{k<\Lambda} c(i\vec{k})\cdot(-i\vec{k}) |\sigma_{i\vec{k}}|^2$$

$$\int d^d x\, (\vec{\sigma}\cdot\vec{\sigma})^2 = \int d^d x \left(\sum_i \sigma_i(\vec{x}) \sigma_i(\vec{x})\right) \left(\sum_j \sigma_j(\vec{x}) \sigma_j(\vec{x})\right)$$

$$= \frac{1}{L^{3d}} \sum_{ij} \sum_{k k' k'' < \Lambda} (\sigma_{i\vec{k}} \sigma_{i\vec{k}'}) (\sigma_{j\vec{k}''} \sigma_{j\vec{k}'''}) \delta(\vec{k}+\vec{k}'+\vec{k}''+\vec{k}''')$$

于是

$$\boxed{\frac{H[\sigma]}{T} = a_0 L^d + \frac{1}{2L^d} \sum_{i=1}^{n} \sum_{k<\Lambda} (a_2 + c k^2) |\sigma_{i\vec{k}}|^2 + \frac{1}{4L^{3d}} \sum_{i,j=1}^{n} \sum_{k,k',k'',k'''<\Lambda} \sigma_{i\vec{k}} \sigma_{i\vec{k}'} \sigma_{j\vec{k}''} \sigma_{j\vec{k}'''} \delta(\vec{k}+\vec{k}'+\vec{k}''+\vec{k}''')}$$

一般记法：$\vec{\sigma}(\vec{x}) \longrightarrow \vec{\phi}(\vec{x})$，$\vec{\sigma}_{\vec{k}} \longrightarrow \vec{\phi}_{\vec{k}}$