第七讲

前面几次课讨论了：统计物理的朗道理论

指出：在描述宏观系统的性质时需要引入对称性的自发破缺的概念。

实验测量可以用 linear response theory 描述，会把系统的响应与平衡系统的关联函数相关。

问题：

有没有方便的方法或语言来计算这些关联函数，或者给我们一些对这些关联函数的直观和定性的理解？

答案是有。这一方法或语言就是路径积分方法

利用这一方法，我们将 $Tr\, e^{-\beta H}$ 表示成对"路径"的求和。每一条路径都有一个权重。路径在 d+1 维"时-空"之中。

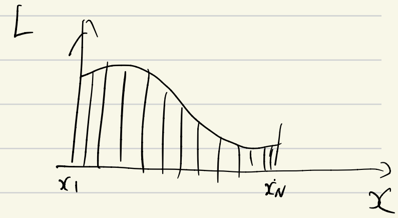
- 权重不一定是实的。当其为实时，对应一个 d+1 维空间中经典模型
- 量子系统有其特有的 Berry phase，带来新的物理。

我们回顾一下经典力学：质点动力学

Lagrangian: $L = p\cdot\dot{x} - H(x,p)$ （以单粒子为例） 另一种表示：$L = T - V = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2x^2$

Action: $S = \int dt\, L(x,\dot{x},p)$

最小作用量原理：$\delta S = 0$. 可导出运动方程.



S 是 L 的泛函 (functional): $F:\ \underset{\downarrow\, L}{f(x)} \in H \longrightarrow \underset{\downarrow\, S}{F} \in R.$

可理解为 $F(f(x_1), f(x_2), \cdots, f(x_N))$ 多元函数. 那么 $\delta F = \sum_i \frac{\partial F}{\partial f(x_i)} \Delta f(x_i)$

$$S = \sum_{t_i} L(x(t_i), \dot{x}(t_i), p(t_i))$$

$$\delta S = \sum_{t_i}\left[\frac{\partial L}{\partial x(t_i)}\Delta x(t_i) + \frac{\partial L}{\partial \dot{x}(t_i)}\Delta\dot{x}(t_i) + \frac{\partial L}{\partial p(t_i)}\Delta p(t_i)\right]$$

$$= \int dt\left(\delta x(t)\frac{\partial L}{\partial x} + \delta\dot{x}(t)\frac{\partial L}{\partial \dot{x}} + \delta p(t)\frac{\partial L}{\partial p}\right)$$

$$\overset{分部}{=} \int dt\left\{\delta x(t)\underbrace{\left(\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}\right)}_{泛函导数\ \frac{\delta S}{\delta x}} + \delta p(t)\underbrace{\frac{\partial L}{\partial p}}_{\frac{\delta S}{\delta p}}\right\}$$ （利用 δx 在 t_i 与 t_f 为零）

任意 $\delta x(t)$ 与 $\delta p(t)$，都有 $\delta S = 0$. $\Rightarrow$ ① $\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0$ 此为拉格朗日方程.

② $\frac{\partial L}{\partial p} = 0$

①与② 可改写为：$\begin{cases} -\frac{\partial H}{\partial x} = \frac{dp}{dt} \\ \frac{dx}{dt} = \frac{\partial H}{\partial p} \end{cases}$ 此为 Hamilton Eq.

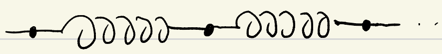
例子：一维谐振子

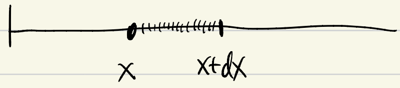
牛顿力学：设 $V(x) = \frac{1}{2}m\omega^2x^2$. $m\ddot{x} = -\frac{\partial V}{\partial x} = -m\omega^2 x$

拉氏方程：$L = p\dot{x} - \frac{p^2}{2m} - \frac{1}{2}m\omega^2x^2$. $\frac{\partial L}{\partial p} = 0 \Rightarrow \dot{x} = \frac{p}{m}$；$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0 \Rightarrow -m\omega^2 x = \frac{dp}{dt} = m\ddot{x}$

哈密顿方程：$\frac{dp}{dt} = -m\omega^2x^2$；$\dot{x} = \frac{p}{m}$ 将 $\dot{x} = \frac{p}{m}$ 代入 L，得 $L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2x^2 = T - V$

推广到多粒子系统：





N个质点用全同弹簧相连，弹性系数为k，y_i为第i个质点对平衡位置的偏离，a是质点平衡距离：

$$L=\frac{1}{2}\sum_i^N\left[m\dot{y}_i^2-k(y_{i+1}-y_i)^2\right]=\sum_i\frac{a}{2}\left[\frac{m}{a}\dot{y}_i^2-ka\left(\frac{y_{i+1}-y_i}{a}\right)^2\right]=\sum_i^N a\mathcal{L}_i$$

$\mathcal{L}_i$ 记是 Lagrangian density：单位长度拉氏量

推广到连续系统：考虑 $dx \gg a\to 0$ $\quad \sum_i a=\int\frac{dx}{a}\cdot a \quad \frac{m\cdot\frac{dx}{a}}{dx}=\frac{m}{a}\to\mu$ 质量密度，

$$\frac{y_{i+1}-y_i}{a}=\frac{y(x+dx)-y(x)}{dx}\longrightarrow\frac{\partial y(x)}{\partial x},\quad ka\longrightarrow Y=\text{Young's modulus},\quad \frac{Y}{2}\left(\frac{\partial y}{\partial x}\right)^2\text{: 单位长度势能}$$

$$L=\int\mathcal{L}\,dx,\quad \text{其中}\ \mathcal{L}=\frac{1}{2}\left[\mu\dot{y}^2-Y\left(\frac{\partial y}{\partial x}\right)^2\right]\longleftarrow\text{拉氏密度}$$

注意 $y_i\to y(x,t)$ 连续空间与时间的函数.

$$\delta S=\delta\int_{t_i}^{t_f}L\,dt=\delta\int_{t_i}^{t_f}dt\int dx\,\mathcal{L}\left(y,\dot{y},\frac{\partial y}{\partial x}\right)$$

$$=\int dt\int dx\left\{\frac{\partial\mathcal{L}}{\partial y}\delta y+\frac{\partial\mathcal{L}}{\partial\left(\partial y/\partial x\right)}\delta\left(\frac{\partial y}{\partial x}\right)+\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial y}{\partial t}\right)}\delta\left(\frac{\partial y}{\partial t}\right)\right\}$$

$$=\int dt\int dx\left\{\frac{\partial\mathcal{L}}{\partial y}\delta y-\frac{\partial}{\partial x}\left(\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial y}{\partial x}\right)}\right)\delta y-\frac{\partial}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial y}{\partial t}\right)}\right)\delta y\right\}$$

这里作了分部积分，利用了在 t_i 与 t_f，x 边界，y 值确定，即 $\delta y=0$

$$\delta S=0\ \Rightarrow\ \boxed{\frac{\partial}{\partial x}\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial y}{\partial x}\right)}+\frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial y}{\partial t}\right)}-\frac{\partial\mathcal{L}}{\partial y}=0}$$

对于我们的弹簧链，记是

$$\boxed{Y\frac{\partial^2 y}{\partial x^2}-\mu\frac{\partial^2 y}{\partial t^2}=0}$$

这是波动方程，波速 $v=\sqrt{Y/\mu}\ \ (=\omega/k)$

$\mathcal{H}$对应 Hamiltonian density：$\mathcal{L}=\dot{y}\frac{\partial\mathcal{L}}{\partial\dot{y}}-\mathcal{H}$，$\frac{\partial\mathcal{L}}{\partial\dot{y}}$ 对应动量.

$$\mathcal{H}=\frac{1}{2}\mu\dot{y}^2+\frac{1}{2}Y\left(\frac{\partial y}{\partial x}\right)^2=T+V$$

推广到二维：mattress（席梦思）.

习惯：$y \to \varphi$，　$\mu \to \sigma$，　$Y \to \rho$

$$\mathcal{L}(\varphi) = \frac{1}{2}\sigma\left(\frac{\partial\varphi}{\partial t}\right)^2 - \frac{1}{2}\rho\left[\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2\right]$$

$$S = \int_{t_i}^{t_f} dt \int d^2x\, \mathcal{L}(\varphi)$$

$$\delta S = 0 \Rightarrow -\sigma\frac{\partial^2\varphi}{\partial t^2} + \rho\left(\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2}\right) = 0$$

即 $\frac{\partial^2\varphi}{\partial t^2} - c^2\nabla^2\varphi = 0$

$c = \sqrt{\rho/\sigma}$ 为声速.

更一般地，考虑 质点之间互作用包括：$-\tau\varphi^2 - \xi\varphi^4 + \cdots$

定义 $x_0 = ct$，$x_1 = x$，$x_2 = y$

$$S = \int d^dx \left[\frac{1}{2}(\partial\varphi)^2 + \frac{1}{2}m^2\varphi^2 - \frac{1}{3!}g\varphi^3 - \frac{1}{4!}\lambda\varphi^4 + \cdots\right]$$

$$(\partial\varphi)^2 = \left(\frac{\partial\varphi}{\partial x_0}\right)^2 - \left(\frac{\partial\varphi}{\partial x}\right)^2 - \left(\frac{\partial\varphi}{\partial y}\right)^2$$

这就是 Landau-Ginzburg 作用量，具有 Lorentz-invariant 即对 t 与 x 同阶的导数.

4维时空：$\mathcal{L} = \frac{1}{2}g_{\mu\nu}\partial^\mu\varphi\partial^\nu\varphi - \frac{1}{2}m^2\varphi^2$

$g = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$ 度规 metric

$$\delta S = 0 \Rightarrow -\frac{\partial\mathcal{L}}{\partial\varphi} + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\right) = 0 \Rightarrow \boxed{\ddot{\varphi} - \nabla\cdot\nabla\varphi + m^2\varphi = 0}$$

简写成 $\boxed{\partial_\mu\partial^\mu\varphi + m^2\varphi = 0}$ $\leftarrow$ Klein-Gorden 方程.

$$\partial_\mu\partial^\mu \equiv g_{\mu\nu}\partial^\mu\partial^\nu$$

约定：1. 重复指标求和 2. $a^\mu \equiv (a^0, \vec{a})$ 反协变4矢，$a_\mu = g_{\mu\nu}a^\nu = (a^0, -\vec{a})$

$a^\mu = g^{\mu\nu}a_\nu$

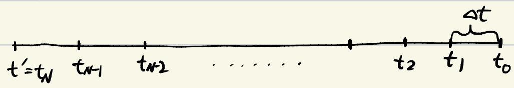
量子力学的路径积分形式:

$$i\hbar\frac{\partial|\psi\rangle}{\partial t} = H|\psi\rangle \Rightarrow |\psi(t')\rangle = \hat U(t',t_0)|\psi(t_0)\rangle, \quad \hat U = e^{-iH(t'-t_0)} \qquad (\hbar=1)$$

$$\psi(x',t') = \langle x'|\psi(t')\rangle = \int\langle x'|\hat U(t',t_0)|x\rangle\langle x|\psi(t_0)\rangle dx = \int \langle x'|\hat U(t',t_0)|x\rangle\, \psi(x,t_0)\,dx$$

我们记 $\boxed{\langle x'|\hat U(t',t_0)|x\rangle = U(x',t';x,t_0)}$，称其为传播子:

将 $t'-t$ 这段时间分成 N 分: $t'-t = N\Delta t$.



$$\hat U = e^{-iH(t'-t_0)} = e^{-iH(t'-t_{N-1})}\cdot e^{-iH(t_{N-1}-t_{N-2})}\cdot\cdots\cdot e^{-iH(t_1-t_0)}$$

$$\langle x'|\hat U|x\rangle = \int \langle x'|e^{-iH\Delta t}|x_{N-1}\rangle\langle x_{N-1}|e^{-iH\Delta t}|x_{N-2}\rangle\langle x_{N-2}|\cdots|x_1\rangle\langle x_1|e^{-iH\Delta t}|x\rangle\, dx_{N-1}\cdot dx_{N-2}\cdots dx_1$$

考虑 $\langle x_{k+1}|e^{-iH\Delta t}|x_k\rangle \approx \langle x_{k+1}|(1-iH\Delta t)|x_k\rangle$ $(\because \Delta t\to 0)$ 这里 k 对应时刻 $t_k = t_0 + k\cdot\Delta t$

其中 $\langle x_{k+1}|H|x_k\rangle = \int\langle x_{k+1}|p_k\rangle\langle p_k|H|x_k\rangle\, dp_k$

由于 $\hat H(\hat x,\hat p)$ 是 $\hat x$ 与 $\hat p$ 的函数. $\hat p|p_k\rangle = p_k|p_k\rangle \Rightarrow \boxed{\langle p_k|\hat H|x_k\rangle = H(x_k,p_k)\langle p_k|x_k\rangle}$

又: $\boxed{\langle x_k|p_k\rangle = \frac{1}{\sqrt{2\pi}}e^{ip_kx_k}, \quad \langle p_k|x_k\rangle = \langle x_k|p_k\rangle^*}$

我们有: $\langle x_{k+1}|e^{-iH\Delta t}|x_k\rangle = \langle x_{k+1}|x_k\rangle - i\Delta t\langle x_{k+1}|H|x_k\rangle = \int dp_k\, \langle x_{k+1}|p_k\rangle\langle p_k|x_k\rangle\,(1-i\Delta t\, H(x_k,p_k))$

$$= \int_{-\infty}^{\infty}\frac{dp_k}{2\pi}\, e^{ip_k(x_{k+1}-x_k) - i\Delta t\, H(x_k,p_k)}$$

最后: $$\langle x'|\hat U|x\rangle = \int\frac{dp_{N-1}\cdots dp_0}{(2\pi)^N}\int dx_{N-1}\cdots dx_1\; e^{i\sum_{k=0}^{N-1}[p_k(x_{k+1}-x_k) - H(x_k,p_k)]},$$

取极限 $N\to\infty$, $\Delta t\to 0$, $x_{k+1}-x_k = \dot x(t)\,dt$

$$\boxed{\langle x'|\hat U|x\rangle = \int_{x(t_0)=x}^{x(t')=x'}\mathcal{D}p(t)\,\mathcal{D}x(t)\; e^{\frac{i}{\hbar}S}, \quad S = \int_{t_0}^{t'}dt\;\; p(t)\dot x(t) - H(x(t),p(t))]}$$

量子力学：从(x,t_0)到(x',t')有无穷多条可能的路径(path). 每条路径的几率幅(或权重)为 $e^{iS/\hbar}$

由于$\hbar$非常小，S的微小变化可导致 $e^{\frac{iS}{\hbar}}$ 的剧烈变化.

但在$\delta S=0$的路径附近，$e^{\frac{iS}{\hbar}}$不变. 因此，这样的路径对积分贡献大，也就是经典力学给出的路径.

- 动量的积分往往可以求出：

利用 $\int_{-\infty}^{\infty} dp\, e^{-\frac{p^2}{2\left(\frac{m\hbar}{i\,dt}\right)}} = \sqrt{2\pi}\left(\frac{m\hbar}{i\,dt}\right)^{\frac{1}{2}}$，配方 $p(t)\dot{x}(t) - \frac{p^2(t)}{2m} = -\frac{(p-m\dot{x})^2}{2m} + \frac{m}{2}\dot{x}^2$

$$\langle x'|\hat{U}|x\rangle = \left(\frac{m}{2\pi i\hbar\Delta t}\right)^{\frac{N}{2}}\int \mathcal{D}x\; e^{\frac{i}{\hbar}S}$$，$S=\int_{t_0}^{t'} dt\left[\frac{1}{2}m\dot{x}^2(t) - V(x(t))\right]$，即通常的 $L=T-V$

重新定义 $\int_x^{x'}\mathcal{D}(x) = \lim_{N\to\infty}\left(\frac{m}{2\pi i\hbar\Delta t}\right)^{\frac{N}{2}}\int dx_{N-1}\cdots dx_1$

$$\langle x'|\hat{U}|x\rangle = \int \mathcal{D}x\; e^{iS/\hbar}$$