第十讲

§2.2 关联函数 correlation functions

LG 超越 Landau 理论之处在于它能告诉我们 $\phi(\vec{x})$ 在空间的变化规律.

即使在基态（没有 domain wall）$\phi(\vec{x})$ 也在变化（偏离 saddle point 解）.

$$\langle\phi(\vec{x})\rangle = \begin{cases} 0 & T>T_c \\ \pm M_0 & T<T_c \end{cases}$$

基态不是一潭死水. 而是很多可能位型的统计求和, 它们将决定基态是仅有平静背景下小小的涟漪, 还是惊涛骇浪足以掩盖 $\langle\phi\rangle$.

这样的空间涨落是由关联函数描述的, 最简单的关联函数是两点关联函数

$\langle\phi(\vec{x})\,\phi(\vec{y})\rangle$: 告诉我们 $\vec{x}$ 的 m 与 $\vec{y}$ 点的 m 怎样关联.

$$\langle\phi(\vec{x})\,\phi(\vec{y})\rangle \equiv \int \mathcal{D}\phi(\vec{x})\quad \phi(\vec{x})\phi(\vec{y})\left(\frac{e^{-\beta F[\phi(\vec{x})]}}{Z}\right), \qquad \langle\phi(\vec{x})\rangle = \frac{\int\mathcal{D}\phi\quad \phi(\vec{x})\, e^{-\beta F[\phi]}}{Z}$$

由于 $\langle\phi(\vec{x})\rangle$ 在 $T>T_c$ 与 $T<T_c$ 不同, 一般计算 connected correlation function:

$$\langle\phi(\vec{x})\phi(\vec{y})\rangle_c \equiv \langle\phi(x)\,\phi(y)\rangle - \langle\phi\rangle^2$$

计算方法: 考虑 $B(\vec{x})$: 在空间变化, LG 自由能修正为:

$$F[\phi(\vec{x})] = \int d^d x\left[\frac{\gamma}{2}(\nabla\phi)^2 + \frac{\mu^2}{2}\phi^2(x) - B(\vec{x})\,\phi(\vec{x})\right]$$

把 Z 理解为 $B(x)$ 的泛函（给我一个函数 $B(x)$, 还你一个 real number Z）

$$Z[B(\vec{x})] = \int\mathcal{D}\phi\quad e^{-\beta F}$$

考虑 $\log Z$ 的泛函导数. （理解: 可把 Z 看成多元函数, 每个 $\vec{x}$ 上的 B 是一个元, 泛函导数→求偏导）

$$\frac{1}{\beta}\frac{\delta\log Z}{\delta B(\vec{x})} = \frac{1}{\beta Z}\frac{\delta Z}{\delta B(\vec{x})} = \frac{1}{Z}\int\mathcal{D}\phi\ \phi(\vec{x})e^{-\beta F} = \langle\phi(\vec{x})\rangle_B$$

$\langle\cdot\rangle_B$ 提醒我们是在磁场 $B(\vec{x})$ 下求期望值, 最终取 $B(\vec{x})=0$, 得到想要的结果 $\langle\phi\rangle$

量化地：通过对$\vec{x}$和$\vec{y}$两点的B求导数，我们得到

$$\frac{1}{\beta^2}\frac{\delta^2 \log Z}{\delta B(\vec{x})\,\delta B(\vec{y})} = \frac{1}{\beta^2 Z}\frac{\delta^2 Z}{\delta B(\vec{x})\,\delta B(\vec{y})} - \frac{1}{\beta^2 Z^2}\frac{\delta Z}{\delta B(\vec{x})}\frac{\delta Z}{\delta B(\vec{y})}$$

$$= \langle \phi(\vec{x})\,\phi(\vec{y})\rangle_B - \langle \phi(\vec{x})\rangle_B \langle \phi(\vec{y})\rangle_B$$

$T>T_c$ 时，$\langle\phi\rangle_{B=0}=0$

$$= \langle \phi(\vec{x})\,\phi(\vec{y})\rangle$$

§2.2.1 The Gaussian Path Integral.

同计算比热一样，我们在傅立叶空间计算. $B(\vec{x}) = \frac{1}{V}\sum_{\vec{k}} e^{i\vec{k}\cdot\vec{x}} B_{\vec{k}} = \int \frac{d^dk}{(2\pi)^d} e^{i\vec{k}\cdot\vec{x}} B_{\vec{k}}$

$$-\int d^dx\, B(\vec{x})\,\phi(\vec{x}) = -\int d^dx \left[\int \frac{d^dk_1}{(2\pi)^d} B_{\vec{k}_1} e^{i\vec{k}_1\cdot\vec{x}}\right]\left[\int \frac{d^dk_2}{(2\pi)^d} \phi_{\vec{k}_2} e^{i\vec{k}_2\cdot\vec{x}}\right]$$

$$= -\int \frac{d^dk}{(2\pi)^d} B_{-\vec{k}}\, \phi_{\vec{k}}$$

因此

$$F[\phi_k] = \int \frac{d^dk}{(2\pi)^d}\left[\frac{1}{2}(\gamma k^2+\mu^2)\,\phi_{\vec{k}}\,\phi_{-\vec{k}} - B_{-\vec{k}}\,\phi_{\vec{k}}\right]$$

定义 $\hat{\phi}_{\vec{k}} = \phi_{\vec{k}} - \frac{B_{\vec{k}}}{\gamma k^2+\mu^2}$ 配方，平移 field.

$$\left(\hat{\phi}_{\vec{k}} + \frac{B_{\vec{k}}}{\gamma k^2+\mu^2}\right)\left(\hat{\phi}_{-\vec{k}} + \frac{B_{-\vec{k}}}{\gamma k^2+\mu^2}\right) = |\hat{\phi}_k|^2 + \frac{B_{\vec{k}}}{\gamma k^2+\mu^2}\hat{\phi}_{-\vec{k}} + \frac{B_{-\vec{k}}}{\gamma k^2+\mu^2}\hat{\phi}_{\vec{k}} + \frac{|B_{\vec{k}}|^2}{(\gamma k^2+\mu^2)^2}$$

$$B_{-\vec{k}}\left(\hat{\phi}_{\vec{k}} + \frac{B_{\vec{k}}}{\gamma k^2+\mu^2}\right) = B_{-\vec{k}}\,\hat{\phi}_{\vec{k}} + \frac{|B_{\vec{k}}|^2}{\gamma k^2+\mu^2}$$

$$\int \frac{d^dk}{(2\pi)^d} B_{-\vec{k}}\,\hat{\phi}_{\vec{k}} = \frac{1}{2}\int \frac{d^dk}{(2\pi)^d} B_{-\vec{k}}\,\hat{\phi}_{\vec{k}} + \frac{1}{2}\int \frac{d^dk}{(2\pi)^d} B_{\vec{k}}\,\hat{\phi}_{-\vec{k}} \qquad (\vec{k}\to-\vec{k} \text{ 不改变 } d^dk)$$

$\Longrightarrow$

$$F[\hat{\phi}_{\vec{k}}] = \int \frac{d^dk}{(2\pi)^d}\left[\frac{1}{2}(\gamma k^2+\mu^2)\,|\hat{\phi}_k|^2 - \frac{1}{2}\frac{|B_k|^2}{\gamma k^2+\mu^2}\right]$$

$$Z = \prod_k{}' \int d\hat{\phi}_k\, d\hat{\phi}_{-k}\, e^{-\beta F[\hat{\phi}_{\vec{k}}]}$$

$$= e^{-\beta F_{thermo}} \exp\left(\frac{\beta}{2}\int \frac{d^dk}{(2\pi)^d}\frac{|B_k|^2}{\gamma k^2+\mu^2}\right)$$

求关联需要回到实空间，作逆变换

$$B_{\vec{k}} = \int d^d x \, e^{-i\vec{k}\cdot\vec{x}} B(\vec{x})$$

代入 Z:

$$Z[B(\vec{x})] = e^{-\beta F_{th}} \exp\left(\frac{\beta}{2}\int d^d x\, d^d y \; B(\vec{x})\, G(\vec{x}-\vec{y})\, B(\vec{y})\right)$$

其中 $$\boxed{G(x-y) = \int \frac{d^d k}{(2\pi)^d} \frac{e^{-i\vec{k}\cdot(\vec{x}-\vec{y})}}{\gamma k^2 + \mu^2}} \qquad ①$$

$$\Rightarrow \boxed{\langle \phi(\vec{x})\phi(\vec{y})\rangle = \frac{1}{\beta^2}\frac{\delta^2 \log Z}{\delta B(\vec{x})\delta B(\vec{y})}\bigg|_{B=0} = \frac{1}{\beta} G(\vec{x}-\vec{y})}$$

注：$\delta B(\vec{x})$ 与 $\delta B(\vec{y})$ 中 $\vec{x}$ 与 $\vec{y}$ 是确定两点，$\int d^d x' d^d y' \, B(\vec{x}') G(\vec{x}'-\vec{y}') B(\vec{y}')$ 中 $\vec{x}'$ 与 $\vec{y}'$ 是积分变量，因此包含 $\vec{x}'=\vec{x}, \vec{y}'=\vec{y}$ 与 $\vec{x}'=\vec{y}, \vec{y}'=\vec{x}$ 两个贡献

取对数后 F_{th} 无贡献，$\log\left[e^{\int d^d x \int d^d y}\right] = \int d^d x \int dy$

剩下的事就是计算傅立叶积分①式。 记 $\vec{x}-\vec{y} \to \vec{x}$

· $G(\vec{x})$ 旋转不变，只与距离 $|x|=r$ 有关。 （对 $\vec{k}\cdot\vec{x}$ 积分，转动 $\vec{x}\to\vec{x}'$，可通过 $\vec{k}\to\vec{k}'$ 消去变化。）

$$G(\vec{x}) = G(r) = \frac{1}{\gamma}\int \frac{d^d k}{(2\pi)^d} \frac{e^{-i\vec{k}\cdot\vec{x}}}{k^2 + \frac{1}{\xi^2}}$$

其中 $\xi^2 \equiv \frac{\gamma}{\mu^2}$ 定义关联长度.

Trick: $$\frac{1}{k^2+\frac{1}{\xi^2}} = \int_0^\infty dt\, e^{-t\left(k^2+\frac{1}{\xi^2}\right)}$$

$$G(r) = \frac{1}{\gamma}\int \frac{d^d k}{(2\pi)^d}\int_0^\infty dt\, e^{-i\vec{k}\cdot\vec{x} - t\left(k^2 + 1/\xi^2\right)}$$

$$= \frac{1}{\gamma}\int \frac{d^d k}{(2\pi)^d}\int_0^\infty dt\, e^{-t\left(\vec{k}+\frac{i\vec{x}}{2t}\right)^2} e^{-r^2/4t - t/\xi^2}$$

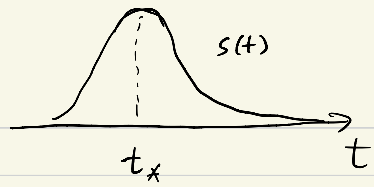
$$= \frac{1}{\gamma(4\pi)^{d/2}}\int_0^\infty dt\, t^{-d/2} e^{-r^2/4t - t/\xi^2}$$

$$-t\left(\vec{k}+\frac{i\vec{x}}{2t}\right)^2 = -tk^2 - i\vec{k}\cdot\vec{x} + \frac{x^2}{4t}$$

$$x^2 \to r^2$$

$$\left(\vec{k}+\frac{i\vec{x}}{2t}\right)^2 = \sum_j \left(k_j + \frac{i x_j}{2t}\right)^2$$

$$\int dk_j\, e^{-t\left(k_j+\frac{i x_j}{2t}\right)^2} = \sqrt{2\pi \frac{1}{2t}}$$



我们先关注 G(r) 的各种极限.

$$G(r)\sim\int_0^\infty dt\, e^{-S(t)} \qquad \text{with} \quad S=\frac{r^2}{4t}+\frac{t}{\xi^2}+\frac{d}{2}\log t$$

使用 saddle point 近似： 在 S(t) 极小处 展开 S(t)，然后 高斯积分

saddle point $t=t_*$ 满足 $S'(t_*)=0$. G(r) 可近似为

$$G(r)\sim\int_0^\infty dt\, e^{-S(t_*)+S''(t_*)\frac{t^2}{2}} = \sqrt{\frac{\pi}{2S''(t_*)}}\, e^{-S(t_*)}$$

由：
$$S'(t)=\frac{-r^2}{4t^2}+\frac{1}{\xi^2}+\frac{d}{2}\frac{1}{t}=0 \Rightarrow t_*=\frac{\xi^2}{2}\left(-\frac{d}{2}+\sqrt{\frac{d^2}{4}+\frac{r^2}{\xi^2}}\right)$$

两个 limits：

- $r\gg\xi$. $\quad t_*\approx\frac{r\xi}{2}$,

$$S(t_*)=\frac{r^2}{4(\frac{r\xi}{2})}+\frac{\frac{r\xi}{2}}{\xi^2}+\frac{d}{2}\log\left(\frac{r\xi}{2}\right)=\frac{r}{\xi}+\frac{d}{2}\log\left(\frac{r\xi}{2}\right)$$

$$S'(t)=\frac{-r^2}{4t^2}+\frac{1}{\xi^2}+\frac{d}{2}\frac{1}{t}\Rightarrow S''=\frac{2r^2}{4t^3}-\frac{d}{2}\frac{1}{t^2}\Rightarrow S''(t_*)=\frac{r^2}{2(\frac{r\xi}{2})^3}-\frac{d}{2}\frac{4}{(r\xi)^2}=\frac{4}{r\xi^3}\quad \left(\text{用到 } \frac{r}{\xi}\gg 1\right)$$

$$G(r)\sim\frac{1}{(r^{-\frac12}\xi^{-\frac32})}\frac{e^{-r/\xi}}{r^{\frac d2}\xi^{\frac d2}}\sim\frac{1}{\xi^{\frac d2-\frac32}}\frac{e^{-r/\xi}}{r^{\frac d2-\frac12}} \qquad r\gg\xi$$

- $r\ll\xi$：

$$t_*=\frac{\xi^2}{2}\left(-\frac d2+\sqrt{\frac{d^2}{4}+\frac{r^2}{\xi^2}}\right)=\frac{\xi^2}{2}\left(-\frac d2+\sqrt{\frac{d^2}{4}\left(1+\frac{4r^2}{d^2\xi^2}\right)}\right)=\frac{\xi^2}{2}\left(-\frac d2+\frac d2\left(1+\frac{2r^2}{d^2\xi^2}\right)\right)\approx\frac{r^2}{2d}$$

$$S=\frac{r^2}{4t}+\frac{t}{\xi^2}+\frac d2\log t\Rightarrow S(t_*)=\frac{r^2}{4(\frac{r^2}{2d})}+\frac{\frac{r^2}{2d}}{\xi^2}+\frac d2\ln\left(\frac{r^2}{2d}\right)=\frac d2+\frac d2\ln\left(\frac{r^2}{2d}\right)$$

$$S''=\frac{2r^2}{4t^3}-\frac d2\frac{1}{t^2}\Rightarrow S''(t_*)=\frac{r^2}{2(\frac{r^2}{2d})^3}-\frac d2\frac{1}{(\frac{r^2}{2d})^2}=\frac{4d^3}{r^4}-\frac{2d^3}{r^4}=\frac{2d^3}{r^4}$$

$$G(r)\sim\frac{1}{(\frac{d^3}{r^4})^{\frac12}}\frac{e^{-\frac d2}}{(\frac{r^2}{2d})^{\frac d2}}\sim\frac{1}{r^{d-2}}$$

—— Ornstein-Zernicke correlation

$d=2$ 有问题