第十六讲

我们先来回顾一下 RG 变换。

$$\text{给定 } F[\phi]=\int d^dx\left[\tfrac{1}{2}\nabla\phi\cdot\nabla\phi+\tfrac{1}{2}r_0\phi^2+g\phi^4+\cdots\right]\longrightarrow \text{参数空间 } \vec{\mu}=(r_0,g,\cdots)$$

1. 粗粒化： $F[\phi]$ 写到动量空间 $F[\phi_k]$

$$\phi_k=\phi_k^- + \phi_k^+,\qquad \phi_k^- \text{ 在 } 0<k<\frac{\Lambda}{\zeta},\quad \phi_k^+ \text{ 在 } \frac{\Lambda}{\zeta}<k<\Lambda.$$

$$F[\phi_k]=F_0[\phi_k^-]+F_0[\phi_k^+]+F_I[\phi_k^-,\phi_k^+]$$

$$Z=\int\mathcal{D}\phi_k\, e^{-F[\phi_k]}=\int_{0<k<\Lambda/\zeta}\prod\phi_k^-\, e^{-F_0[\phi_k^-]}\int_{\frac{\Lambda}{\zeta}<k<\Lambda}\prod\phi_k^+\, e^{-F_0[\phi_k^+]-F_I[\phi_k^-,\phi_k^+]}=\int_{0<k<\Lambda/\zeta}\prod\phi_k^-\, e^{-F'[\phi_k^-]}$$

$$\text{对于 } F_G=\int d^dx\,\tfrac{1}{2}(\nabla\phi\cdot\nabla\phi+r_0\phi^2)=\int_0^\Lambda\frac{d^dk}{(2\pi)^d}\,\tfrac{1}{2}(k^2+r_0)|\phi_k|^2,\qquad \vec{\mu}=r_0\,\hat{e}_t$$

$$F'[\phi_k^-]=\int_0^{\Lambda/\zeta}\frac{d^dk}{(2\pi)^d}\,\tfrac{1}{2}(k^2+r_0')|\phi_k^-|^2=\int d^dx\left[\tfrac{1}{2}\nabla\phi^-\cdot\nabla\phi^-+\tfrac{r_0'}{2}(\phi^-)^2\right],\qquad r_0'=r_0$$

$$\text{一般情况：}\quad F'[\phi^-]=\int d^dx\left[\tfrac{1}{2}\nabla\phi^-\cdot\nabla\phi^-+\tfrac{1}{2}r_0'(\phi^-)^2+g'(\phi^-)^4+\cdots\right]$$

2. rescale：

$$k'=\zeta k,\quad x'=\frac{x}{\zeta},\quad \phi'_{k'}=\zeta^{\Delta\phi_k}\phi_k,\quad \phi'(x')=\zeta^{\Delta\phi}\phi(x).\quad \Delta\phi_k=\Delta\phi-d$$

$$F'[\phi]\to F_\zeta[\phi']=\int d^dx'\left[\tfrac{1}{2}\nabla_{x'}\phi'(x')\cdot\nabla_{x'}\phi'(x')+\tfrac{1}{2}r_0(\zeta)\phi'(x')\phi'(x')+g(\zeta)(\phi'(x'))^4+\cdots\right]$$

这就定义了： $\vec{\mu}\to\vec{\mu}(\zeta)=R_\zeta\vec{\mu}$

对于 F_G， $\vec{\mu}(\zeta)=r_0(\zeta)\vec{e}_t=\zeta^2 r_0\vec{e}_t$

3. 不动点与附近的分析：

不平凡不动点： $\vec{\mu}^*$ 对应 $\xi=\infty$. 在 $\vec{\mu}^*$ 附近线性化 R_ζ

$$\vec{\mu}=\vec{\mu}^*+\delta\vec{\mu},\qquad \text{线性化 } \delta\vec{\mu}'=R_\zeta^L\,\delta\vec{\mu}$$

$$\delta\vec{\mu}=\sum_j t_j\vec{e}_j,\qquad \delta\vec{\mu}'=\sum_j t_j\zeta^{y_j}\vec{e}_j,\qquad \frac{1}{\nu}=y_1$$

$$\text{对 } F_G,\quad t_1\to r_0,\quad y_1=2\quad\Rightarrow\quad\frac{1}{\nu}=\frac{1}{2}.$$

$$\text{此外：自治要求 } \Delta\phi_k=-1-\frac{d}{2}\;\Rightarrow\;\Delta\phi=\frac{d-2}{2}\;\Rightarrow\;\eta=0.$$

现在考虑含 ϕ^4 项 in LG:

$$F[\phi] = \int d^d x \, [\underbrace{\tfrac{1}{2}\nabla\phi\cdot\nabla\phi + \tfrac{1}{2} r_0 \phi^2}_{F_G} \underbrace{+ g_0 \phi^4}_{F_I}] = \int_0^\Lambda \frac{d^d k}{(2\pi)^d} \left[\tfrac{1}{2}(k^2+r_0)|\phi_k|^2 + g_0 \phi_{k_1}\phi_{k_2}\phi_{k_3}\phi_{k_4} (2\pi)^d \delta(\vec{k}_1+\vec{k}_2+\vec{k}_3+\vec{k}_4) \right]$$

用到 $\int d^d x \, e^{i k\cdot x} = (2\pi)^d \delta^d \vec{k}$

(1). 粗粒化: $\phi_{\vec{k}} = \phi_{\vec{k}}^- + \phi_{\vec{k}}^+$

$$F[\phi] = F_0[\phi^-] + F_0[\phi^+] + F_I[\phi^-, \phi^+] \leftarrow \text{选取 } F_G \text{ 为 } F_0$$

其中 $F_0 = \int \frac{d^d k}{(2\pi)^d} \frac{1}{2}(k^2+r_0)\phi_k \phi_{-k} = \int_0^{\Lambda/3} \frac{d^d k}{(2\pi)^d} \frac{1}{2}(k^2+r_0)\phi_k^- \phi_{-k}^- + \int_{\Lambda/3}^{\Lambda} \frac{d^d k}{(2\pi)^d} \frac{1}{2}(k^2+r_0)\phi_k^+ \phi_{-k}^+$

$$F_I = \int d^d x \, g_0 \phi^4 = \int \frac{d^d k}{(2\pi)^d} g_0 \phi_{k_1}\phi_{k_2}\phi_{k_3}\phi_{k_4}(2\pi)^d \delta(\vec{k}_1+\vec{k}_2+\vec{k}_3+\vec{k}_4)$$

$$Z = \int \prod_{k<\Lambda} \phi_{\vec{k}} \, e^{-F} = \int \prod_{k<\Lambda/3} d\phi_k^- \left[e^{-F_0[\phi^-]} \cdot \int \prod_{\frac{\Lambda}{3}<k<\Lambda} d\phi_k^+ \, e^{-F_0[\phi^+] - F_I[\phi_k^-, \phi_k^+]} \right] = \int \prod_{k<\Lambda/3} d\phi_k^- \, e^{-F'[\phi^-]}$$

$$e^{-F'[\phi^-]} = e^{-F_0[\phi^-]} \cdot \int \prod_{\frac{\Lambda}{3}<k<\Lambda} d\phi_k^+ \, e^{-F_0[\phi^+] - F_I[\phi_k^-, \phi_k^+]}$$

$$= e^{-F_0[\phi^-]} \cdot \frac{\int \mathcal{D}\phi_k^+ \, e^{-F_0[\phi_k^+]} \cdot e^{-F_I[\phi_k^-, \phi_k^+]}}{\int \mathcal{D}\phi_k^+ \, e^{-F_0[\phi_k^+]}} \cdot \int \mathcal{D}\phi_k^+ \, e^{-F_0[\phi_k^+]}$$

$$= e^{-F_0[\phi_k^-]} \langle e^{-F_I[\phi_k^-, \phi_k^+]} \rangle_+ \, e^{-A}$$

定义了: $e^{-A} = \int \mathcal{D}\phi_k^+ \, e^{-F_0[\phi_k^+]} \rightarrow$ 高斯分布给出的归一化因子

于是:

$$\boxed{F'[\phi^-] = F_0[\phi_k^-] - \ln \langle e^{-F_I[\phi_k^-, \phi_k^+]} \rangle_+ - A}$$

求助于微扰来计算 $\langle e^{-F_I} \rangle_+$

无量纲化: 设 $L_0 \equiv r_0^{-\frac{1}{2}}$ $\quad \varphi \equiv \frac{\phi}{L_0^{1-d/2}}, \quad \vec{r} \equiv \frac{\vec{x}}{L_0}, \quad \bar{g}_0 \equiv \frac{g_0}{L_0^{d-4}}$

$$F_G = \int d^d r \left[\tfrac{1}{2}(\nabla\varphi)^2 + \tfrac{1}{2}\varphi^2 \right], \quad F_I = \int d^d r \, \bar{g}_0 \varphi^4$$

当 $\bar{g} = g_0 r_0^{\frac{d-4}{2}} \ll 1$, 可以微扰展开 e^{-F_I}.

$$\ln \langle e^{-F_I} \rangle_+ = \ln \langle 1 - F_I + \tfrac{1}{2} F_I^2 + \cdots \rangle_+ = \ln(1 - \langle F_I \rangle_+ + \tfrac{1}{2}\langle F_I^2 \rangle_+ + \cdots)$$

再利用 $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots$, 设 $x = -\langle F_I \rangle_+ + \frac{1}{2}\langle F_I^2 \rangle_+ - \cdots$

$x^2 = \langle F_I \rangle_+^2 - \frac{1}{2}\langle F_I \rangle_+ \langle F_I^2 \rangle_+$

$$\Rightarrow \boxed{\ln \langle e^{-F_I} \rangle_+ = -\langle F_I \rangle_+ + \tfrac{1}{2}\left[\langle F_I^2 \rangle_+ - \langle F_I \rangle_+^2 \right] + \cdots}$$

也写作: $\langle e^{-F_I} \rangle_+ = e^{-\langle F_I \rangle_+ + \frac{1}{2}[\langle F_I^2 \rangle_+ - \langle F_I \rangle_+^2] + \cdots}$ 称累积展开.

• $\langle F_I \rangle_+$ 是微扰的一阶项（g_0 的一阶）.

观察：$F_I[\phi_k^-, \phi_k^+] = g_0 \int \prod_{i=1}^{4} \frac{d^d k_i}{(2\pi)^d} \phi_{k_1}\phi_{k_2}\phi_{k_3}\phi_{k_4} (2\pi)^d \delta^d(\sum_i \vec{k}_i)$

每个 $\phi_k = \phi_k^- + \phi_k^+$，$\phi_{k_1}\phi_{k_2}\phi_{k_3}\phi_{k_4}$ 共 16 项，可以分成 5 类

i) $\phi_{k_1}^- \phi_{k_2}^- \phi_{k_3}^- \phi_{k_4}^-$，记作 $\langle F_I \rangle_+^{----}$ 不含任何 ϕ_k^+，贡献 $g_0 \int d^d x (\phi^-)^4$ 给 $F'[\phi]$

ii) $4\ \phi_{k_1}^- \phi_{k_2}^- \phi_{k_3}^- \phi_{k_4}^+$，一个 ϕ_k^+ 其 $\frac{\Lambda}{3} < k < \Lambda$，

$$\langle F_I \rangle_+^{---+} = g_0 \int \prod_i \frac{d^d k_i}{(2\pi)^d} \phi_{k_1}^- \phi_{k_2}^- \phi_{k_3}^- \langle \phi_{k_4}^+ \rangle_+ (2\pi)^d \delta^d(\sum_i \vec{k}_i)$$

$\langle \phi_{k_4}^+ \rangle_+ = \frac{1}{N} \int_{-\infty}^{\infty} d\phi_{k_4}^+ \, \phi_{k_4}^+ e^{-\frac{a}{2}|\phi_k|^2} = 0$ 在 Gaussian 积分后消失

iii) $6\ \phi_{k_1}^- \phi_{k_2}^- \phi_{k_3}^+ \phi_{k_4}^+$：重要. $6 = C_4^2$. k_3 与 k_4 在 $\Lambda/3$ 与 Λ 之间，$\langle F_I \rangle_+^{--++}$，需计算 $\langle \phi_{k_3}^+ \phi_{k_4}^+ \rangle_+$

iv) $4\ \phi_{k_1}^- \phi_{k_2}^+ \phi_{k_3}^+ \phi_{k_4}^+$，cubic in ϕ_k^+，消失 $\because \langle \phi_{k_2}^+ \phi_{k_3}^+ \phi_{k_4}^+ \rangle_+ = 0$

v) $\phi_{k_1}^+ \phi_{k_2}^+ \phi_{k_3}^+ \phi_{k_4}^+$，不含 ϕ_k^-，给出 free energy 中常数. 不重要

• 计算：

$$\langle F_I \rangle_+^{--++} = 6 g_0 \int \prod_{i=1}^{4} \frac{d^d k_i}{(2\pi)^d} \phi_{k_1}^- \phi_{k_2}^- \langle \phi_{k_3}^+ \phi_{k_4}^+ \rangle_+ (2\pi)^d \delta^d(\sum_i \vec{k}_i)$$

如果 k_3 与 k_4 独立：$\langle \phi_{k_3}^+ \phi_{k_4}^+ \rangle_+ = 0$，$\because \langle \phi_{k_3}^+ \phi_{k_4}^+ \rangle_+ = \langle \phi_{k_3}^+ \rangle_+ \langle \phi_{k_4}^+ \rangle_+ = 0$. 两个独立高斯积分.

当 $\vec{k}_3 = -\vec{k}_4$ 即 $\phi_{k_3}^+ = (\phi_{k_4}^+)^*$，积分才不为零.

$$\langle \phi_q^+ \phi_{-q}^+ \rangle_+ = \frac{\int d\phi_q \, d\phi_q^* \, |\phi_q^+|^2 e^{-\frac{1}{2}(r_0+q^2)|\phi_q|^2}}{\int d\phi_q \, d\phi_q^* \, e^{-\frac{1}{2}(r_0+q^2)|\phi_q|^2}} = \frac{1}{r_0+q^2}$$

用到高斯积分：$\frac{\int_{-\infty}^{\infty} dx \, x^2 e^{-\frac{ax^2}{2}}}{\int_{-\infty}^{\infty} dx \, e^{-\frac{ax^2}{2}}} = \frac{1}{a}$

也可由高斯路径分公式：$\langle \phi(x)\phi(y) \rangle = G(x-y)$，$G(x-y) = \int \frac{d^d k}{(2\pi)^d} \frac{e^{-i\vec{k}\cdot(\vec{x}-\vec{y})}}{r_0 + k^2}$ 得到.

写成 $\vec{k}$ 连续形式：
$$\boxed{\langle \phi_{k_3}^+ \phi_{k_4}^+ \rangle_+ = (2\pi)^d \delta^d(\vec{k}_3 + \vec{k}_4) G_0(\vec{k}_3), \quad G_0(\vec{k}_3) = \frac{1}{r_0 + k_3^2}}$$

最终：
$$\langle F_I \rangle_+^{--++} = 6 g_0 \int \prod_{i=1}^{4} \frac{d^d k_i}{(2\pi)^d} \phi_{k_1}^- \phi_{k_2}^- G_0(\vec{k}_3) (2\pi)^d \delta^d(\vec{k}_3 + \vec{k}_4) (2\pi)^d \delta^d(\vec{k}_1 + \vec{k}_2 + \vec{k}_3 + \vec{k}_4)$$

$$= 6 g_0 \int_0^{\Lambda/3} \frac{d^d k}{(2\pi)^d} \phi_{\vec{k}}^- \phi_{-\vec{k}}^- \int_{\Lambda/3}^{\Lambda} \frac{d^d q}{(2\pi)^d} \frac{1}{r_0 + q^2}$$

$\langle F_I \rangle_+^{--++}$ 与 $F_0[\phi^-] = \int_0^{\Lambda/\zeta} \frac{d^d k}{(2\pi)^d} \frac{1}{2}(r_0 + k^2)\, \phi^-_{\vec{k}} \phi^-_{-\vec{k}}$ 合并.

$$\Rightarrow \boxed{r_0' = r_0 + 12 g_0 \int_{\Lambda/\zeta}^{\Lambda} \frac{d^d q}{(2\pi)^d} \frac{1}{r_0 + q^2}}$$

RG 第二步: rescale: $\vec{k} \to \vec{k}' = \zeta \vec{k}$, $\phi^-_{\vec{k}} \to \phi'_{\vec{k}'} = \zeta^{\Delta\phi_k} \phi^-_{\vec{k}}$

$$F'[\phi^-] = \int_0^{\Lambda/\zeta} \frac{d^d k}{(2\pi)^d} \frac{1}{2}(r_0' + k^2)\, \phi^-_{\vec{k}} \phi^-_{-\vec{k}}$$

$$= \int_0^{\Lambda} \frac{d^d k'}{(2\pi)^d} \zeta^{-d} \frac{1}{2}(r_0' + \zeta^{-2} k'^2) \zeta^{-2\Delta\phi_k} \phi'_{\vec{k}'} \phi'_{-\vec{k}'}$$

- k'^2 系数为 1 $\rightarrow \zeta^{-2\Delta\phi_k - 2 - d} = \zeta^0 \Rightarrow \Delta\phi_k = -1 - \frac{d}{2}$
- $$\boxed{r_0(\zeta) = r_0' \zeta^{-2\Delta\phi_k - d} = \zeta^2 \left(r_0 + 12 g_0 \int_{\Lambda/\zeta}^{\Lambda} \frac{d^d q}{(2\pi)^d} \frac{1}{r_0 + q^2} \right)} \quad (A)$$
- $r_0 = 0$ 不再"不动"了. 因为有了来自 ϕ^4 的贡献.

- 再计算 $\langle F_I \rangle^{----} = g_0 \int_0^{\Lambda/\zeta} \frac{d^d k_1}{(2\pi)^d} \cdot \frac{d^d k_2}{(2\pi)^d} \cdot \frac{d^d k_3}{(2\pi)^d} \cdot \frac{d^d k_4}{(2\pi)^d}\, \phi^-_{k_1} \phi^-_{k_2} \phi^-_{k_3} \phi^-_{k_4} (2\pi)^d \delta^d(\sum_i \vec{k}_i)$

构成 $F'[\phi]$ 中的 ϕ^4 项

rescale 之后 $= g_0 \int_0^{\Lambda} \prod_i \frac{d^d k_i'}{(2\pi)^d} \zeta^{-4d - 4\Delta\phi_k} \phi'_{\vec{k}_1'} \phi'_{\vec{k}_2'} \phi'_{\vec{k}_3'} \phi'_{\vec{k}_4'} (2\pi)^d \delta(\sum_i \vec{k}_i')$

$= g_0 \zeta^{-3d - 4(-1-\frac{d}{2})} \int d^d x\, \phi^4(x) \qquad (\delta(\vec{k}') = \zeta^{-d} \delta(\vec{k}))$

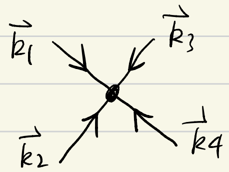
即: $\boxed{g_0(\zeta) = g_0 \zeta^{4-d}}$ (B)

(A) 与 (B) 给出 $R_\zeta \vec{\mu} = \vec{\mu}(\zeta)$

④

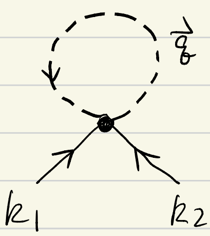
Feynman Diagrams 来帮忙.

$$\langle F_I\rangle_+^{----} = g_0 \int_0^{\Lambda/\zeta} \frac{d^dk_1}{(2\pi)^d}\frac{d^dk_2}{(2\pi)^d}\frac{d^dk_3}{(2\pi)^d}\frac{d^dk_4}{(2\pi)^d}\, \phi^-_{\vec k_1}\phi^-_{\vec k_2}\phi^-_{\vec k_3}\phi^-_{\vec k_4}\,(2\pi)^d\,\delta^d\Big(\sum_i \vec k_i\Big) = g_0\int d^d\vec x\,(\phi^-)^4$$



1. 外腿 (external leg): $\phi^-_{\vec k}$,
2. $\bullet = g_0(2\pi)^d\,\delta(\vec k_1+\vec k_2+\vec k_3+\vec k_4)$ 顶角 (vertex), 可理解为动量守恒
3. 对每一条 external leg 作积分. $\int_0^{\Lambda/\zeta}\frac{d^dk}{(2\pi)^d}$

$$\langle F_I\rangle_+^{++--} = g_0\int_0^{\Lambda/\zeta}\frac{d^dk_1}{(2\pi)^d}\frac{d^dk_2}{(2\pi)^d}\,\phi^-_{\vec k_1}\phi^-_{\vec k_2}\,(2\pi)^d\delta(\vec k_1+\vec k_2)\int_{\Lambda/\zeta}^{\Lambda}\frac{d^dq}{(2\pi)^d}\frac{1}{r_0+q^2}$$



1. 两条腿 $\phi^+_{\vec k_3}$ 和 $\phi^+_{\vec k_4}$ 作平均后相连，形成内线 (internal line)
内线携带动量 $\vec q$ 和因子 $G_0(\vec q)$, 形成圈 (loop)
并积分 $\int_{\Lambda/\zeta}^{\Lambda}\frac{d^dq}{(2\pi)^d}$
2. 顶角. $\bullet = g_0(2\pi)^d\,\delta(\vec k_1+\vec k_2)$