第十七讲

现在考虑含 ϕ^4 项 in LG:

$$F[\phi] = \int d^dx\,[\underbrace{\tfrac{1}{2}\nabla\phi\cdot\nabla\phi + \tfrac{1}{2}r_0\phi^2}_{F_G=F_0} + \underbrace{g_0\phi^4}_{F_I}] = \int_0^\Lambda \frac{d^dk}{(2\pi)^d}\left[\tfrac{1}{2}(k^2+r_0)|\phi_k|^2\right] + g_0\int\prod_i \frac{d^dk_i}{(2\pi)^d}\phi_{k_1}\phi_{k_2}\phi_{k_3}\phi_{k_4}(2\pi)^d\delta^d(\sum_i k_i)]$$

(1). 粗粒化: $\phi_{\vec{k}} = \phi_k^- + \phi_k^+$

$$F[\phi] = F_0[\phi^-] + F_0[\phi^+] + F_I[\phi^-,\phi^+]$$

其中 $$F_0 = \int\frac{d^dk}{(2\pi)^d}\,\tfrac{1}{2}(k^2+r_0)\phi_k\phi_{-k} = \underbrace{\int_0^{\Lambda/s}\frac{d^dk}{(2\pi)^d}\,\tfrac{1}{2}(k^2+r_0)\phi_k^-\phi_{-k}^-}_{F_0[\phi^-]} + \underbrace{\int_{\Lambda/s}^{\Lambda}\frac{d^dk}{(2\pi)^d}\,\tfrac{1}{2}(k^2+r_0)\phi_k^+\phi_{-k}^+}_{F_0[\phi^+]}$$

$$F_I = \int d^dx\; g_0\phi^4 = \int\prod_{i=1}^4\frac{d^dk_i}{(2\pi)^d}\,g_0\,\phi_{k_1}\phi_{k_2}\phi_{k_3}\phi_{k_4}(2\pi)^d\delta(\vec{k}_1+\vec{k}_2+\vec{k}_3+\vec{k}_4)$$

$$Z = \int\prod_{k<\Lambda}\phi_{\vec{k}}\; e^{-F} = \int\prod_{k<\Lambda/s}d\phi_k^-\left[e^{-F_0[\phi^-]}\cdot\int\prod_{\Lambda/s\le k<\Lambda}d\phi_k^+\; e^{-F_0[\phi^+]-F_I[\phi_k^-,\phi_k^+]}\right] = \int\prod_{k<\Lambda/s}d\phi_k^-\; e^{-F'[\phi]}$$

$$e^{-F'[\phi]} = e^{-F_0[\phi^-]}\cdot\frac{\int\mathcal{D}\phi_k^+\,e^{-F_0[\phi_k^+]}\cdot e^{-F_I[\phi_k^-,\phi_k^+]}}{\int\mathcal{D}\phi_k^+\,e^{-F_0[\phi_k^+]}}\cdot\int\mathcal{D}\phi_k^+\,e^{-F_0[\phi_k^+]}$$

$$= e^{-F_0[\phi_k^-]}\langle e^{-F_I[\phi_k^-,\phi_k^+]}\rangle_+\; e^{-A}$$

于是:

$$\boxed{F'[\phi^-] = F_0[\phi_k^-] - \ln\langle e^{-F_I[\phi_k^-,\phi_k^+]}\rangle_+ - A}$$

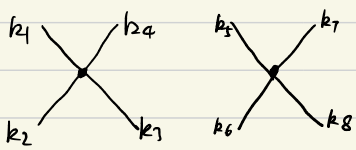
$$\boxed{\ln\langle e^{-F_I}\rangle_+ = -\langle F_I\rangle_+ + \tfrac{1}{2}\left[\langle F_I^2\rangle_+ - \langle F_I\rangle_+^2\right] + \cdots}$$ 累积展开.

$$F'[\phi^-] = F_0[\phi^-] + \langle F_I\rangle_+ - \tfrac{1}{2}\left[\langle F_I^2\rangle_+ - \langle F_I\rangle_+^2\right] + \cdots - A.$$

下面计算 $-\frac{1}{2}(\langle F_I^2\rangle_+ - \langle F_I\rangle_+^2)$ 对 $F'[\phi]$ 的贡献.

从坐标空间看: $F_I^2 = \int d^dx\,d^dy\; g_0^2\,\phi^4(x)\,\phi^4(y)$,

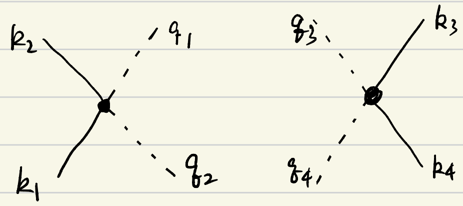
$$= g_0\int\prod_{i=1}^8\frac{d^dk_i}{(2\pi)^d}\;\phi_{k_1}\phi_{k_2}\phi_{k_3}\phi_{k_4}(2\pi)^d\delta(\sum_{i=1}^4 k_i)\;\phi_{k_5}\phi_{k_6}\phi_{k_7}\phi_{k_8}(2\pi)^d\delta(\sum_{j=5}^8 k_j)$$



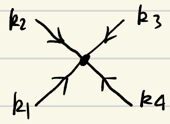
每个 $\phi_k = \phi_k^- + \phi_k^+$, 共有 $2^8=256$ 种情况

考虑下面这种情况

$$g_0^2 \int_0^{\Lambda/s} \left[\prod_{i=1}^{4} \frac{d^d k_i}{(2\pi)^d}\, \phi^-_{k_i}\right] \int_{\Lambda/s}^{\Lambda} \left[\prod_{j=1}^{4} \frac{d^d q_j}{(2\pi)^d}\right] \langle \phi^+_{q_1} \phi^+_{q_2} \phi^+_{q_3} \phi^+_{q_4} \rangle_+ (2\pi)^{2d}\, \delta^d(\vec{k}_1+\vec{k}_2+\vec{q}_1+\vec{q}_2)\, \delta^d(\vec{q}_3+\vec{q}_4+\vec{k}_3+\vec{k}_4)$$



$\Rightarrow$ 对 $g_0 \int d^dx\, \phi^4$ 有贡献.



关键: 计算 $\langle \phi^+_{q_1} \phi^+_{q_2} \phi^+_{q_3} \phi^+_{q_4} \rangle_+$, 利用 Wick's theorem

引理: 对于n个高斯变量 ϕ_a,

$$\langle f(\phi) \rangle = \frac{1}{N} \int_{-\infty}^{\infty} d^n\phi\; f(\phi)\; e^{-\frac{1}{2}\phi \cdot G^{-1} \cdot \phi}$$

G 是 $n\times n$ 可逆实对称矩阵, $N = [\det(2\pi G)]^{\frac{1}{2}}$

那么对任意常数 B^a, 有

$$\boxed{\langle e^{B_a \phi_a} \rangle = e^{\frac{1}{2} B_a \langle \phi_a \phi_b \rangle B_b}} \qquad (1)$$

证明. 由广义高斯积分 (参见第1讲)

$$\langle e^{B_a \phi_a} \rangle = e^{\frac{1}{2} B_a G_{ab} B_b}$$

定义 $Z = \int d^n\phi\; e^{-\frac{1}{2}\phi G^{-1} \phi - B_a \phi_a} \Rightarrow \langle \phi_a \phi_b \rangle = \frac{\partial^2}{\partial B_a \partial B_b} \ln Z \Big|_{B=0} \Rightarrow \langle \phi_a \phi_b \rangle = G_{ab}$.

$$\therefore \langle e^{B_a \phi_a} \rangle = e^{\frac{1}{2} B_a \langle \phi_a \phi_b \rangle B_b}$$

Taylor 展开 (1) 式:

$$\langle e^{B_a \phi_a} \rangle = 1 + B_a \langle \phi_a \rangle + \frac{1}{2} B_a B_b \langle \phi_a \phi_b \rangle + \frac{1}{3!} B_a B_b B_c \langle \phi_a \phi_b \phi_c \rangle + \frac{1}{4!} B_a B_b B_c B_d \langle \phi_a \phi_b \phi_c \phi_d \rangle + \cdots$$

$$e^{\frac{1}{2} B_a \langle \phi_a \phi_b \rangle B_b} = 1 + \frac{1}{2} B_a B_b \langle \phi_a \phi_b \rangle + \frac{1}{8} B_a B_b B_c B_d \langle \phi_a \phi_b \rangle \langle \phi_c \phi_d \rangle + \cdots$$

对比左右两端同幂次项,

$$\langle \phi_{a_1} \phi_{a_2} \cdots \phi_{a_l} \rangle = 0, \quad \text{for } l \text{ odd}. \qquad (2)$$

再看偶次幂项：

$$\langle\phi_a\phi_b\phi_c\phi_d\rangle = \langle\phi_a\phi_b\rangle\langle\phi_c\phi_d\rangle+\langle\phi_a\phi_c\rangle\langle\phi_b\phi_d\rangle+\langle\phi_a\phi_d\rangle\langle\phi_b\phi_c\rangle$$

$$\frac{1}{24}\sum_{abcd} B_aB_bB_cB_d\langle\phi_a\phi_b\phi_c\phi_d\rangle$$

与 $\frac18 B_aB_bB_cB_d\langle\phi_a\phi_b\rangle\langle\phi_c\phi_d\rangle = \frac18\cdot\frac13\Big(\sum_{ab}B_a\langle\phi_a\phi_b\rangle B_b\sum_{cd}B_c\langle\phi_c\phi_d\rangle B_d + \sum_{ac}B_a\langle\phi_a\phi_c\rangle B_c\sum_{bd}B_b\langle\phi_b\phi_d\rangle B_d$

$+\sum_{ad}B_a\langle\phi_a\phi_d\rangle B_d\sum_{bc}B_b\langle\phi_b\phi_c\rangle B_c\Big)$

$$= \frac{1}{24}B_aB_bB_cB_d\left[\langle\phi_a\phi_b\rangle\langle\phi_c\phi_d\rangle+\langle\phi_a\phi_c\rangle\langle\phi_b\phi_d\rangle+\langle\phi_a\phi_d\rangle\langle\phi_b\phi_c\rangle\right]\Big)$$

可见：

$$\boxed{\langle\phi_{a_1}\phi_{a_2}\cdots\phi_{a_{2l}}\rangle = \langle\phi_{a_1}\phi_{a_2}\rangle\cdots\langle\phi_{a_{2l-1}}\phi_{a_{2l}}\rangle+\cdots\cdots}$$ （共 $(2l-1)!!$ 项）

这就是 Wick's theorem.

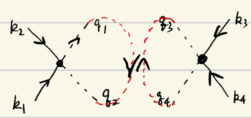
• 利用 Wick 定理来计算 $\langle\phi^+_{q_1}\phi^+_{q_2}\phi^+_{q_3}\phi^+_{q_4}\rangle_+$

$$\langle\phi^+_{q_1}\phi^+_{q_2}\phi^+_{q_3}\phi^+_{q_4}\rangle_+ = \langle\phi^+_{q_1}\phi^+_{q_2}\rangle_+\langle\phi^+_{q_3}\phi^+_{q_4}\rangle_+ + \langle\phi^+_{q_1}\phi^+_{q_3}\rangle_+\langle\phi^+_{q_2}\phi^+_{q_4}\rangle_+ + \langle\phi^+_{q_1}\phi^+_{q_4}\rangle_+\langle\phi^+_{q_2}\phi^+_{q_3}\rangle_+$$

已知 $\boxed{\langle\phi^+_q\phi^+_{q'}\rangle_+ = (2\pi)^d\delta^d(q+q')G_0(q)}$ —— (A)

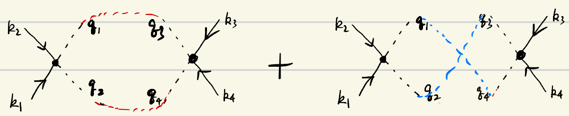
分类讨论： 注意：new delta functions 与原有 δ functions 的组合.

• $\int\prod_{j=1}^4 d^dq_j\ \langle\phi^+_{q_1}\phi^+_{q_2}\rangle_+\langle\phi^+_{q_3}\phi^+_{q_4}\rangle_+\ \delta^d(k_1+k_2+q_1+q_2)\,\delta^d(k_3+k_4+q_3+q_4)$ 利用(A)



$$\sim\int d^dq_2\,d^dq_4\ G_0(q_2)G_0(q_4)\,\delta^d(k_1+k_2)\,\delta^d(k_3+k_4)$$

这是一个 nolocal 的积分，最终被 $\langle F_I^2\rangle$ 中相应的项消去 ⟶ $\langle F_I\rangle^{--++}\cdot\langle F_I\rangle^{--++}$



• 有意思的是 $\langle\phi^+_{q_1}\phi^+_{q_3}\rangle\langle\phi^+_{q_2}\phi^+_{q_4}\rangle_+$ 与 $\langle\phi^+_{q_1}\phi^+_{q_4}\rangle_+\langle\phi^+_{q_2}\phi^+_{q_3}\rangle_+$

$\int\prod_{j=1}^4 d^dq_i\ \langle\phi^+_{q_1}\phi^+_{q_3}\rangle_+\langle\phi^+_{q_2}\phi^+_{q_4}\rangle_+\ \delta^d(k_1+k_2+q_1+q_2)\,\delta^d(k_3+k_4+q_3+q_4)$ 利用(A)

$\sim\int d^dq_1\,d^dq_2\ G_0(q_1)G_0(q_2)\,\delta^d(k_1+k_2+q_1+q_2)\,\delta^d(k_3+k_4-q_1-q_2)\ \sim\int d^dq\ G_0(q_1)G_0(|k_1+k_2+q_1|)\,\delta^d(k_1+k_2+k_3+k_4)$

公式中只有一个 δ function，给出实空间对 free energy 的 local 贡献. 不会被 $\langle F_I\rangle^2$ 消去：没有连接的图.

计算 $\frac{1}{2}\langle F_I^2\rangle$ 中这一部分: $\binom{4}{2}^2 g_0^2\int_0^{\Lambda/s}\left[\prod_{i=1}^{4}\frac{d^dk_i}{(2\pi)^d}\phi^-_{k_i}\right] f(k_1+k_2)(2\pi)^d\delta^d\left(\sum_{i=1}^{4}k_i\right)$ (2)

- 除掉 $\frac{1}{2}\to 1$, 由于 Wick 定理贡献两项.
- $$f(k_1+k_2)=\int_{\Lambda/s}^{\Lambda}\frac{d^dq}{(2\pi)^d}\frac{1}{q^2+r_0}\;\frac{1}{(k_1+k_2+q)^2+r_0}=\int_{\Lambda/s}^{\Lambda}\frac{d^dq}{(2\pi)^d}\;\frac{1}{(q^2+r_0)^2}\left(1+O(k_1,k_2)\right)$$

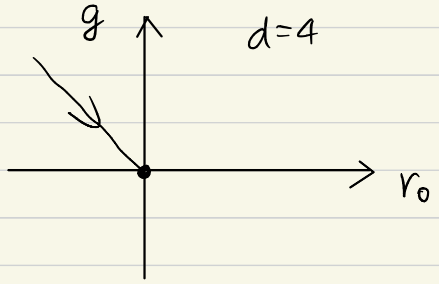
用到:

$$\frac{1}{(k_1+k_2+q)^2+r_0}=\frac{1}{(k_1+k_2)^2+2(k_1+k_2)\cdot q+q^2+r_0}=\frac{1}{r_0+q^2}\cdot\frac{1}{1+\frac{(k_1+k_2)^2+2(k_1+k_2)\cdot q}{r_0+q^2}}$$

- 所有依赖外部 k_1 和 k_2 的项 都会生成 形如 $k^2(\phi^-)^4\sim(\phi^-)^2\nabla^2(\phi^-)^2$ 的贡献. 属于 非关键.
- (2) 是我们需要的, 效果: $\begin{array}{ccc} k_1 & & k_3\\ & \times & \\ k_2 & & k_4\end{array}$

$$\boxed{g_0\to g_0'=g_0-36g_0^2\int_{\Lambda/s}^{\Lambda}\frac{d^dq}{(2\pi)^d}\frac{1}{(q^2+r_0)^2}}$$

尤为重要: 决定了 $d=4$ 时 $g\phi^4$ 项的命运: $d\geq 4$ 时, 由于 rescale: $g'=s^{4-d}g_0$, 上式表明 g' 减少



g 的 RG flow 非常慢, 称 marginally irrelevant

很多情况都是这样: 在考虑 perturbative correction 之后, marginal 变成 marginally irrelevant or marginally relevant

Kondo effect and non-Abelian gauge theory

§ Feynman diagrams

随着展开阶数升高，可能的项数开始爆涨.

$$\boxed{F'[\phi^-] = F_0[\phi^-] + \langle F_I\rangle_+ - \frac{1}{2}\left[\langle F_I^2\rangle_+ - \langle F_I\rangle_+^2\right] + \cdots}$$

假设要计算上式中形如 $g_0^p(\phi^-)^n(\phi^+)^l$ 的一项，可以将其表示 Feynman diagrams.

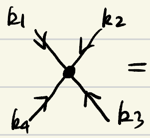
规则

- each ϕ_k^- 由实线表示
- each ϕ_k^+ 由 dotted line 表示
- dotted lines 相互联接形成内圈．或者按所有可能配对，反映 Wick 定理的收缩. 没有单独的 dotted line ∵ l even
- 每个 g_0 因子由一个 vertex 表示（四条线相汇）
- 每条线有一个动量相伴, which is conserved as we move around the diagram.

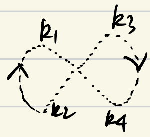
每个图实际是一个积分的简写. 写出如下:

- 每条内部线对应插入一个传播子 $\langle\phi_k^+\phi_{k'}^+\rangle_+ = (2\pi)^d\delta^d(k+k')\,G_0(k)$
- 每个内部圈，对应一个积分 $\int\frac{d^dq}{(2\pi)^d}$
- 每个 vertex 给出一个 power of $g_0(2\pi)^d\delta^d(\sum_i k_i)$，$\delta\to$ 动量守恒，约定 all momenta 都是 incoming
- symmetry factors $\longrightarrow$ 数值系数

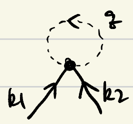
最终得到 $F'[\phi^-]$ 中的一项 $g_0^p(\phi^-)^n \longrightarrow$ n个外线，p个顶角.

at order g_0, $g_0\phi^-_{k_1}\phi^-_{k_2}\phi^-_{k_3}\phi^-_{k_4} \longrightarrow$ 

$= g_0\int\prod_{i=1}^{4}\frac{d^dk_i}{(2\pi)^d}(2\pi)^d\delta^d(\sum k_i) = g_0\int d^dx\,(\phi^-)^4 \times 1$ tree diagram

$g_0\phi^+_{k_1}\phi^+_{k_2}\phi^+_{k_3}\phi^+_{k_4} \longrightarrow$ 

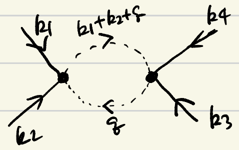
$g_0\int\frac{d^dk_1}{(2\pi)^d}\int\frac{d^dk_3}{(2\pi)^d}G_0(k_1)\,G_0(k_3)(2\pi)^d\delta^d(k_1+k_3)\times 1 =$ 常数.

$g_0\phi^-_{k_1}\phi^-_{k_2}\phi^+_q\phi^+_{-q} \longrightarrow$ 

$= 6g_0\int_{\Lambda/3}^{\Lambda}\frac{d^dq}{(2\pi)^d}\frac{1}{q^2+r_0}\int\frac{d^dk}{(2\pi)^d}\phi^-_k\phi^-_{-k} \longrightarrow g'\int d^dx\,(\phi^-)^2$

loop diagram

at order g_0^2:

$$g_0^2\,(\phi^-)^4(\phi^+)^4 \longrightarrow$$ 

对称系数 $(C_4^2)^2=36$.

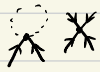
两条内线形成一个loop: $\displaystyle\int\frac{d^d q}{(2\pi)^d}\frac{1}{r_0+q^2}\cdot\frac{1}{r_0+(q+k_1+k_2)^2}=f(k_1,k_2)$

两个δ + 两个传播子:

$$(2\pi)^d\delta(k_1+k_2+q_1+q_2)\cdot(2\pi)^d\delta(k_3+k_4-q_1-q_2)=(2\pi)^{2d}\delta(k_1+k_2+k_3+k_4)\,\delta(q_2-(q_1+k_1+k_2))$$

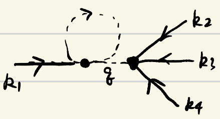
最终形式 $\displaystyle\longrightarrow \binom{4}{2}^2 g_0^2\int_0^{\Lambda/s}\left[\prod_{i=1}^{4}\frac{d^d k_i}{(2\pi)^d}\,\phi^-_{k_i}\right] f(k_1+k_2)(2\pi)^d\delta^d\Big(\sum_{i=1}^4 k_i\Big)\longrightarrow g_0^2\int(\phi^-)^4 d^dx$

由于 $\boxed{F'[\phi]=F_0[\phi]+\langle F_I\rangle_+-\frac{1}{2}\left[\langle F_I^2\rangle_+-\langle F_I\rangle_+^2\right]+\cdots}$ 的累积展开性质:

- 只需考虑 connected graphs. $\langle F_I^2\rangle_+$ 中  与 $\langle F_I\rangle_+^2$ cancell.

- Quantum Field theory 语言: 视 Feynman diagrams 中的线为粒子的世界线

More diagrams



vanishing, 一个 ϕ^- 动量要等于中间的 ϕ^+ 传播子的动量, 但这是不可能的 ϕ^- 与 ϕ^+ 的动量不相等