第十八讲

1. Beta Functions

RG: 1. Kadanoff 变换(粗粒化): $\phi_k = \phi_k^- + \phi_k^+$, 分别属于 $[0, \frac{\Lambda}{\zeta})$ 与 $(\frac{\Lambda}{\zeta}, \Lambda]$

$$F'[\phi^-] = F_0[\phi^-] + \langle F_I\rangle_+ - \frac{1}{2}[\langle F_I^2\rangle_+ - \langle F_I\rangle_+^2] + \cdots$$

$$\text{Gauss} \quad + \quad g_0 \quad + \quad g_0^2$$

2. rescale:

$F'[\phi] \to F[\phi]$, 但是 $\vec{\mu} \longrightarrow \vec{\mu}_\zeta = R_\zeta \vec{\mu}$　　参数空间 $\vec{\mu} = (r_0, g_0, \cdots)$

方便起见，记 $\zeta = e^s$，因此 $\Lambda/\zeta = \Lambda e^{-s} = \Lambda'$，RG 变换写成一阶微分方程的形式，比如：

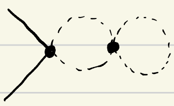
$$\frac{dg}{ds} = \beta(g)$$

历史原因，右侧称为 beta function

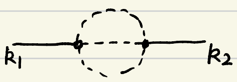
我们下面关注两个最重要的 couplings，r_0 和 g_0，准确到 g_0 的二阶：

$$\boxed{\begin{aligned} r(\zeta) &= \zeta^2(r_0 + a g_0), & a &= 12\int_{\Lambda/\zeta}^{\Lambda}\frac{d^dq}{(2\pi)^d}\frac{1}{q^2+r_0} \\ g(\zeta) &= \zeta^{4-d}(g_0 - b g_0^2) & b &= 36\int_{\Lambda/\zeta}^{\Lambda}\frac{d^dq}{(2\pi)^d}\frac{1}{(q^2+r_0)^2} \end{aligned}}$$

两个没讨论的图，都有两个 loops.


$$= g_0^2\int\frac{d^dk}{(2\pi)^d}\frac{1}{2}C(\Lambda)\,\phi_k^-\phi_{-k}^-$$

导致 a shift in the quadratic term　　$r_0' = r_0 + a g_0 + g_0^2 C(\Lambda)$



$$= g_0^2\int\frac{d^dk}{(2\pi)^d}\frac{1}{2}A(k,\Lambda)\,\phi_k^-\phi_{-k}^-$$

重要的是 $A(k,\Lambda)$ 是外部动量 k 的函数，意味着它有两个 relevant 效应.

- $A(k) = A(0) + \frac{1}{2}k^2 A''(0) + \cdots$,
  - 对 r_0 的 RG，依赖于 $A(0,\Lambda)$　　• 还有一个对梯度项的贡献.

$$F'[\phi] = \int d^dx\,\frac{1}{2}\gamma'\nabla\phi^-\cdot\nabla\phi^- + \cdots \quad \text{with} \quad \gamma' = 1 - 2g_0^2A''(0,\Lambda)$$

要求: a new rescaling of the field, 记 $\gamma' = \zeta^\eta$

$$F' \to F_\zeta = \int d^dx'\,\zeta^d\,\frac{1}{2}\zeta^\eta\zeta^{-2}\nabla_{x'}\phi'\nabla_{x'}\phi'\,\zeta^{-2\Delta_\phi} \Rightarrow 2\Delta_\phi = d-2+\eta$$

(动量空间求导差化)

最后这一步称为 "field renormalisation": gives rise to the anomalous dimension of ϕ: η

对 $r(\zeta)$ 与 $g(\zeta)$ 求导，可得到 beta functions，包含两项：

第一项分别是 $2r$ 和 $(4-d)g$，来自 r 和 g 的标度维数.

第二项来自 corrections，由 a 与 b 给出

具体过程：考虑 s 为小量，$e^{-s}\approx 1-s$，$\Lambda-\Lambda e^{-s}$ 很小：

$$\int_{\Lambda e^{-s}}^{\Lambda} dq\, f(q) \approx (\Lambda-\Lambda e^{-s}) f(\Lambda) \approx \Lambda f(\Lambda) s \Rightarrow \boxed{\frac{d}{ds}\int_{\Lambda e^{-s}}^{\Lambda} dq\, f(q) = \Lambda f(\Lambda)}$$

- 考虑 $d=4$，$\frac{\Omega_3}{(2\pi)^4} = \frac{1}{8\pi^2}$　　Ω_{d-1} 是 d 维单位球面 S^{d-1} 的面积　$\boxed{\Omega_{d-1} = \frac{2\pi^{d/2}}{\Gamma(d/2)}}$

$$\frac{\Omega_{d-1}}{(2\pi)^d} = \frac{1}{2^{d-1}\pi^{\frac{d}{2}}}\frac{1}{\Gamma(d/2)},\quad \Gamma(n)=(n-1)!,\quad \Gamma(2)=1,\quad \frac{\Omega_{4-1}}{(2\pi)^4}=\frac{1}{2^3\pi^2}$$

例：$\Gamma(\frac{3}{2})=\frac{\sqrt{\pi}}{2}$　$\frac{\Omega_2}{(2\pi)^3} = \frac{1}{2^2\pi^{\frac{3}{2}}\cdot\frac{\pi^{\frac{1}{2}}}{2}} = \frac{1}{2\pi^2}$，3D球面积 = 4π.

$$\boxed{\frac{dr}{ds} = 2r + \frac{3g}{2\pi^2}\frac{\Lambda^4}{\Lambda^2+r}}$$

由 $r(\zeta)=\zeta^2(r_0+a g_0)$

推导：
$$\frac{dr}{ds} = 2r + \zeta^2 g_0\frac{da}{ds} = 2r + \zeta^2 g_0\, 12\frac{d}{ds}\left[\int_{\Lambda e^{-s}}^{\Lambda}\frac{dq}{8\pi^2}\frac{q^3}{q^2+r_0}\right]$$

$$= 2r + \frac{3g_0\zeta^2}{2\pi^2}\frac{\Lambda^4}{\Lambda^2+r_0} = 2r + \frac{3g}{2\pi^2}\frac{\Lambda^4}{\Lambda^2+r}$$

（用到 $g=g_0$，$r=r_0$，$\zeta^2=1$，∵ s 小量）

$$\boxed{\frac{dg}{ds} = -\frac{db}{ds}g^2 = \frac{-9}{2\pi^2}\frac{\Lambda^4}{(\Lambda^2+r)^2}g^2}$$

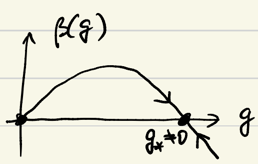
- beta function for r 有两项，第一项来自RG中第2步：重标，第二项来自RG的第一步粗粒化.
- beta function for g 只有一项，没有 g 的线性项，因为 g 在重标中是 marginal 的，

  但是将有来自 g^2 阶 RG 的贡献（线性），它是负的，因此是 marginally irrelevant

- 当 $d<4$，

$\frac{dg}{ds}$ 要加上 $(4-d)g$，成为 relevant，我们将离开 Gaussian fixed point，但是我们不知道将将去哪里，也许后来 $\beta(g)$ 变成小于零？

$d<4$ 时 $\beta(g)$ 的示意图：



当 $g<g_*\neq 0$，RG令 $g\uparrow$

当 $g>g_*\neq 0$，RG令 $g\downarrow$

(2)

2. The Epsilon Expansion ($d=4-\epsilon$)

我们考虑 $d=4-\epsilon$. 且 $\epsilon \ll 1$. 对 RG 方程 (β function) 推广到 从 $d\in Z^+ \Rightarrow d\in R$.

$$\boxed{\begin{aligned} \frac{dr}{ds} &= 2r + \frac{12\Omega_{d-1}}{(2\pi)^d}\frac{\Lambda^4}{\Lambda^2+r}\tilde{g} \\ \frac{d\hat{g}}{ds} &= \epsilon\hat{g} - \frac{36\Omega_{d-1}}{(2\pi)^d}\frac{\Lambda^4}{(\Lambda^2+r)^2}\hat{g}^2 \end{aligned}}$$

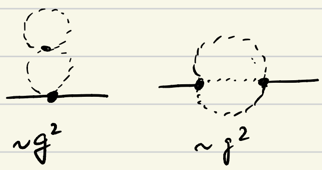
用到 $\int \frac{d^d q}{(2\pi)^d} = \int \frac{dq}{(2\pi)^d} \cdot q^d \cdot \Omega_{d-1}$ $\quad \Lambda^d = \Lambda^4 \Lambda^{-\epsilon}$

$$\frac{d\varsigma^{4-d}}{ds} = \frac{d e^{(4-d)s}}{ds} = (4-d)e^{\epsilon s} = \epsilon \varsigma^{4-d}$$

定义了 $\tilde{g} = \Lambda^{-\epsilon} g$, 多了一个 $\tilde{g}$ 的线性项, 第二式两边同乘 $\Lambda^{-\epsilon}$,

此时 $g \sim \epsilon$, ∵ $d=4$ 时只有一个不动点.

- 我们忽略了两圈图只贡献 ϵ^2 阶修正给 r, 与 beta function 的截断一致.



- $\Omega_{d-1} = \frac{2\pi^{d/2}}{\Gamma(d/2)}$ 包含对 Ω_{d-1} 的 ϵ 阶修正, $2\pi^{\frac{4-\epsilon}{2}} = 2e^{\frac{4-\epsilon}{2}\ln\pi} = 2e^{2\ln\pi} \cdot e^{-\frac{\epsilon\ln\pi}{2}} = 2\pi^2 \cdot (1-\frac{\epsilon}{2}\ln\pi)$

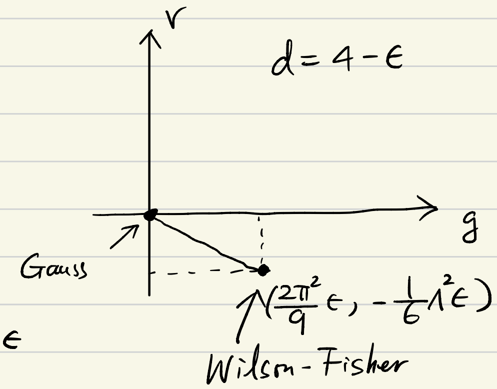
由于 $g\sim\epsilon$, simply use $\Omega_3 = 2\pi^2$, $(2\pi)^d \sim (2\pi)^4$,

$$\boxed{\begin{aligned} \frac{dr}{ds} &= 2r + \frac{3}{2\pi^2}\frac{\Lambda^4}{\Lambda^2+r}\tilde{g} \\ \frac{d\tilde{g}}{ds} &= \epsilon\tilde{g} - \frac{9}{2\pi^2}\frac{\Lambda^4}{(\Lambda^2+r)^2}\tilde{g}^2 \end{aligned}} \qquad (A)$$

这些 beta 函数有 2 个 fixed points

1. Gaussian: $r_*^G = \tilde{g}_*^G = 0$.
2. 新不动点:

$$r_* = -\frac{3}{4\pi^2}\frac{\Lambda^4}{\Lambda^2+r_*}\tilde{g}_*, \qquad \tilde{g}_* = \frac{2\pi^2}{9}\frac{(\Lambda^2+r_*)^2}{\Lambda^4}\epsilon$$



在 ϵ 一阶:

$$r_* = -\frac{3}{4\pi^2}\Lambda^2\left(1-\frac{r_*}{\Lambda^2}\right)\tilde{g}_* \approx -\frac{3}{4\pi^2}\Lambda^2\tilde{g}_* \quad (\text{设 } r_*\sim\epsilon, \text{ 且 } g_*\sim\epsilon)$$

同理: $\boxed{\tilde{g}_* = \frac{2\pi^2}{9}\epsilon \Rightarrow r_* = -\frac{1}{6}\Lambda^2\epsilon}$

这就是著名的 Wilson-Fisher fixed point.

- ϵ 小, $\Rightarrow$ $\tilde{g}_*$ 也是小量, 计算是自洽的.

3. The Wilson-Fisher Fixed point 的物理意义：

为理解 W-F f. p. 附近的RG flows，我们设 $r = r_* + \delta r$，$\tilde{g} = \tilde{g}_* + \delta\tilde{g}$

线性化 beta functions

$$\boxed{\frac{d}{ds}\begin{pmatrix}\delta r\\ \delta\tilde{g}\end{pmatrix} = \begin{pmatrix}2-\epsilon/3, & \frac{3}{2\pi^2}\Lambda^2(1+\frac{\epsilon}{6})\\ 0 & -\epsilon\end{pmatrix}\begin{pmatrix}\delta r\\ \delta\tilde{g}\end{pmatrix}} \qquad (B)$$

推导.

$$\frac{d(r_*+\delta r)}{ds} = 2(r_*+\delta r) + \frac{3}{2\pi^2}\frac{\Lambda^4}{\Lambda^2+(r_*+\delta r)}(\tilde{g}_*+\delta\tilde{g})$$

$$= 2r_* + \frac{3}{2\pi^2}\left[\frac{\Lambda^4}{\Lambda^2+r_*} + \frac{-\Lambda^4}{(\Lambda^2+r_*)^2}\delta r\right]\left[\tilde{g}_* + \delta\tilde{g}\right] + 2\delta r$$

$$\frac{d\delta r}{ds} = 2\delta r - \frac{3}{2\pi^2}\frac{\Lambda^4\tilde{g}_*}{(\Lambda^2+r_*)^2}\delta r + \frac{3}{2\pi^2}\frac{\Lambda^4}{\Lambda^2+r_*}\delta\tilde{g} = \left[2-\frac{3}{2\pi^2}\frac{\Lambda^4\tilde{g}_*}{(\Lambda^2+r_*)^2}\right]\delta r + \frac{3}{2\pi^2}\frac{\Lambda^4}{\Lambda^2+r_*}\delta\tilde{g}$$

代入：$r^* = -\frac{1}{6}\Lambda^2\epsilon$，$\tilde{g}^* = \frac{2\pi^2}{9}\epsilon$，

$$\frac{d\delta r}{ds} = \left[2-\frac{3}{2\pi^2}\cdot\frac{\Lambda^4\frac{2\pi^2}{9}\epsilon}{(\Lambda^2-\frac{1}{6}\Lambda^2\epsilon)^2}\right]\delta r + \frac{3}{2\pi^2}\frac{\Lambda^4}{\Lambda^2(1-\frac{1}{6}\epsilon)}\delta\tilde{g} = (2-\frac{\epsilon}{3})\delta r + \frac{3}{2\pi^2}\Lambda^2(1+\frac{\epsilon}{6})\delta\tilde{g}$$

由：$\frac{d\tilde{g}}{ds} = \epsilon\tilde{g} - \frac{9}{2\pi^2}\frac{\Lambda^4}{(\Lambda^2+r)^2}\tilde{g}^2$

$$\frac{d(\tilde{g}^*+\delta\tilde{g})}{ds} = \epsilon(\tilde{g}_*+\delta\tilde{g}) - \frac{9}{2\pi^2}\frac{\Lambda^4}{(\Lambda^2+r_*+\delta r)^2}(\tilde{g}_*+\delta\tilde{g})^2 = \epsilon\tilde{g}_* - \frac{9\Lambda^4}{2\pi^2}\left[\frac{1}{(\Lambda^2+r_*)^2(1+\frac{2\delta r}{\Lambda^2+r_*})}\right](\tilde{g}_*^2+2\delta\tilde{g}\,\tilde{g}_*)$$

$$= \epsilon(\tilde{g}_*+\delta\tilde{g}) - \frac{9\Lambda^4}{2\pi^2}\frac{1}{(\Lambda^2+r_*)^2}\left[1-\frac{2\delta r}{(\Lambda^2+r_*)}\right](\tilde{g}_*^2+2\delta\tilde{g}\,\tilde{g}_*)$$

利用 $\tilde{g}_* = \frac{2\pi^2}{9}\frac{(\Lambda^2+r_*)^2}{\Lambda^4}\epsilon$，即 $\frac{9}{2\pi^2}\frac{\Lambda^4}{(\Lambda^2+r_*)^2} = \frac{\epsilon}{\tilde{g}_*}$，加上 $\tilde{g}_* \sim \epsilon$

$$\frac{d\delta\tilde{g}}{ds} = \epsilon(\tilde{g}_*+\delta\tilde{g}) - \frac{\epsilon}{\tilde{g}_*}\left[1-\frac{2\delta r}{\Lambda^2(1-\frac{\epsilon}{6})}\right]\tilde{g}_*(\tilde{g}_*+2\delta\tilde{g})$$

$$= \epsilon(\tilde{g}_*+\delta\tilde{g}) - \epsilon(\tilde{g}_*+2\delta\tilde{g}) = -\epsilon\,\delta\tilde{g}$$

下面分析线性化变换 (B).

右端为一个矩阵. 形如 $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$

求解其本征方程: $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\begin{pmatrix} \phi_1^{(i)} \\ \phi_2^{(i)} \end{pmatrix} = \lambda_i \begin{pmatrix} \phi_1^{(i)} \\ \phi_2^{(i)} \end{pmatrix}$

$$\Rightarrow \lambda_1 = 2-\frac{\epsilon}{3}+O(\epsilon^2), \quad \vec{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \lambda_2 = -\epsilon + O(\epsilon^2) \qquad \vec{e}_2 = \begin{pmatrix} \frac{b}{c-a} \\ 1 \end{pmatrix}$$

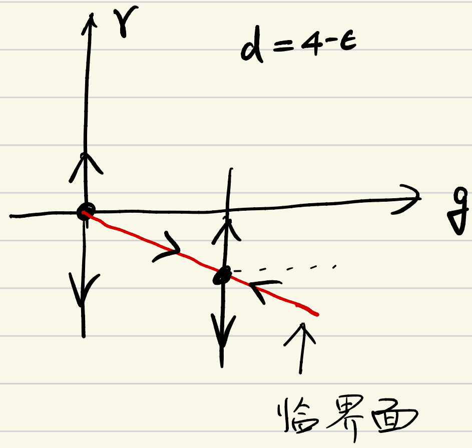
展开:

$$\begin{pmatrix} \delta r \\ \delta g \end{pmatrix} = t_1 \vec{e}_1 + t_2 \vec{e}_2, \quad \xRightarrow{\text{由}(B)} \quad \frac{d}{ds} t_1 \vec{e}_1 + \frac{dt_2}{ds}\vec{e}_2 = \lambda_1 t_1 \vec{e}_1 + \lambda_2 t_2 \vec{e}_2$$

$$\Rightarrow \quad \frac{dt_1}{t_1} = \lambda_1 ds \Rightarrow \ln t_1' = \lambda_1 s + \ln t_1 \Rightarrow t_1' = \zeta^{\lambda_1} t_1$$

(运动方程: $s=0$ 时 $\zeta=1$, $t_1'=t_1$)

λ_1 就是之前定义的 y_1. relevant.



$$\frac{dt_2}{t_2} = \lambda_2 ds \Rightarrow t_2' = \zeta^{\lambda_2} t_2,$$

λ_2 就是 y_2, irrelevant

我们有3个整体的图像对 $d<4$ 的 RG flows,

1. 系统稳定在 Wilson-Fisher Fixed point 其它的 couplings 都是 irrelevant 的.

2. 现在假设我们增加 ϵ, W-F f.p. 移向较大的 g, 微扰的图像失效. 尽管如此, 定性的图像仍正确

3. 由于 W-F fixed point 只有一个关涉的方向, 沿着 $\vec{e}_2$ 方向形成临界面.

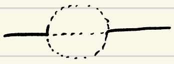
我们只调 $T\to T_c$, 系统处于临界面, 在 RG 变换下, 最终将抵达这个不动点

到达 W.F. 的"速度"由 $y_2=-\epsilon$ 控制

离开W-F f.p.的关键方向是温度 $t=|T-T_c|$,

$$\nu = \frac{1}{y_1} = \frac{1}{2} + \frac{\epsilon}{12} + O(\epsilon^2)$$

利用超标度关系 $\alpha = 2-d\nu = -\frac{\epsilon}{3}$

其它的临界指数需要η, 与  有关, $\eta=\frac{\epsilon^2}{6}$, 需要计算到 ϵ^2.

在 ϵ 一阶, $\eta=0$.

$$\Delta_\phi = \frac{d-2+\eta}{2} \approx \frac{d-2}{2} = \frac{4-\epsilon-2}{2} = 1-\frac{\epsilon}{2}$$

利用其它标度关系.

$$\beta = \frac{1}{2}-\frac{\epsilon}{6}, \quad \gamma = 1+\frac{\epsilon}{6}, \quad \delta = 3+\epsilon$$

我们真正的兴趣是 $d=3$, $\epsilon=1$, 代入

	α	β	γ	δ	η	ν
MF	0	$\frac{1}{2}$	1	3	0	$\frac{1}{2}$
$\epsilon=1$	$2-4\cdot\frac{7}{12}=\frac{1}{3}$	$\frac{1}{3}$	$\frac{7}{6}$	4	0	$\frac{7}{12}$
$d=3$.	0.11	0.3264	1.2371	4.7898	0.0363	0.6300

α 似应从 $\alpha+2\beta+\gamma=2$ 得到为 $\frac{\epsilon}{6}$

从 $\alpha=2-d\nu$ 则得 $-\frac{\epsilon}{3}$