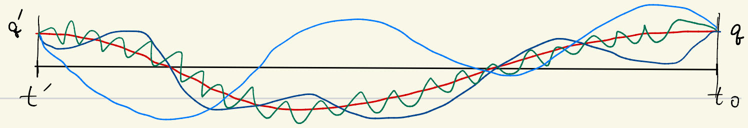
第七讲



红色路径为经典路径.

§. 路径积分与统计物理

前面我们导出了初态为 $|q\rangle$，经过 t' 时间 $(t_0=0)$ 达到 $|q'\rangle$ 态的几率幅可表成路径积分

$$\langle q'|e^{-iHt'}|q\rangle=\int \mathcal{D}q\, e^{iS}, \quad S=\int_0^{t'}dt\left[\frac{1}{2}m\dot{q}^2(t)-V(q(t))\right], \text{ 被积 } L=T-V \qquad ①$$

定义 $it'=\beta,\ idt=d\tau$ 考虑 $q'=q$，则上式化为

$$\boxed{\langle q|e^{-\beta H}|q\rangle=\int\mathcal{D}q\, e^{-S(q(\tau))}} \qquad ②$$

其中

$$\boxed{S(q(\tau))=-iS'(q(t))=-\int_0^\beta d\tau\left[\frac{1}{2}m\left(\frac{dq}{-id\tau}\right)^2-V(q(\tau))\right]=\int_0^\beta d\tau\left[\frac{1}{2}m\dot{q}^2+V(q(\tau))\right]} \qquad ③$$

注意被积函数变成了 $H=T+V$，但是 $\dot{q}=\frac{dq}{d\tau}$

设该谐振子处于环境温度 T，其 density operator $\rho=\frac{e^{-\beta H}}{Z}$，

其中 $Z=\mathrm{Tr}\, e^{-\beta H}$，可以写成

$$Z=\int dq\,\langle q|e^{-\beta H}|q\rangle, \quad \text{其中 } \beta=\frac{1}{T}$$

被积函数就是②式。将②中 $\mathcal{D}q=\lim_{N\to\infty}\left(\frac{m}{2\pi i\hbar\Delta t}\right)^{\frac{N}{2}}dq_{N-1}\cdots dq_1$ 扩展到包含 dq

则

$$\boxed{Z=\int\mathcal{D}q\, e^{-S[q(\tau)]}} \qquad ④$$

推导②式和③式时，$\beta=it'$ 是实是虚由 t' 是实是虚决定

如果 t' 是取纯虚数，则 β 为正的实数。映射 $t'\to -i\beta$ 也被称为 Wick rotation.

这样统计物理通过路径积分与量子力学(量子场论)建立了一一对应!

附:

$$\frac{1}{2}(e^{it}+e^{-it})=\cos(t)\xrightarrow[\text{or } t=-i\beta]{\text{记 } it=\beta}\frac{1}{2}(e^{\beta}+e^{-\beta})=\cosh(\beta)$$

t 实数　　　　　　　　β 实数

• 我们也可以从 $\langle q'|e^{-\beta H}|q\rangle$ 出发，将 β 分成 N 分，导出

$$\boxed{\langle q'|e^{-\beta H}|q\rangle = \int \mathcal{D}q \int \mathcal{D}p\, e^{-S'},\quad S'(\tau) = \int_0^\beta d\tau \left[-i\, p(\tau)\frac{dq(\tau)}{d\tau} + H(q(\tau), p(\tau))\right]} \qquad \hbar = 1. \quad (5)$$

重要：Berry phase，Topology

再令 $\beta = it'$，$d\tau = i\,dt$，得到之前得到的

$$\boxed{\langle q'|\hat{U}|q\rangle = \int \mathcal{D}p(t)\mathcal{D}q(t)\; e^{iS},\quad S(t) = \int_0^{t'} dt\,[p(t)\dot{q}(t) - H(q(t), p(t))]} \qquad (6)$$

(5) 式积掉 p，我们得到 (2) 和 (3) 式．进而 (4) 式．

在量子场论中计算几率幅（从初态 $|I\rangle$ 到末态 $|F\rangle$，设 $t_0 = 0$）

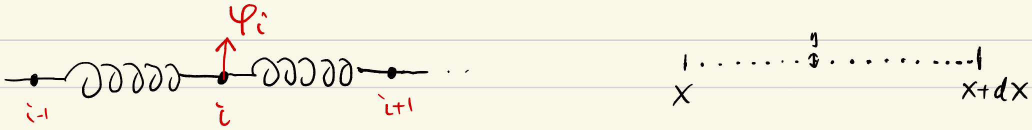
$$\langle F|e^{iHt'}|I\rangle = \int dq_F\, dq_I\, \langle F|q_F\rangle \langle q_F|e^{-iHt'}|q_I\rangle\langle q_I|I\rangle$$

往往人们关心 $|F\rangle = |I\rangle = |0\rangle \leftarrow$ 基态，且 $t' \to \infty$ 即 $\langle 0|e^{-iHt'}|0\rangle$，$t' \to \infty$.

考虑零温，$\beta = \infty$，则 $Z = \mathrm{Tr}\, e^{-\beta H} = \sum_E \langle E|e^{-\beta H}|E\rangle = \langle 0|e^{-\beta H}|0\rangle$

因此在量子场论中，$\langle 0|e^{-iHt'(\to\infty)}|0\rangle$ 也被定义为 Z.

多粒子系统的路径积分



N个质点用全同弹簧相连，弹性系数为k，φ_i为第i个质点对平衡位置的偏离，a是质点平衡距离：

$$L=\frac{1}{2}\sum_i^N\left[m\dot{\varphi}_i^2-k(\varphi_{i+1}-\varphi_i)^2\right]=\sum_i\frac{a}{2}\left[\frac{m}{a}\dot{\varphi}_i^2-ka\left(\frac{\varphi_{i+1}-\varphi_i}{a}\right)^2\right]=\sum_i^N a\mathcal{L}_i$$

推广到连续系统：考虑 $dx \gg a\to 0$，$\sum_i a=\int\frac{dx}{a}\cdot a$，$\frac{m\cdot\frac{dx}{a}}{dx}=\frac{m}{a}\to\sigma$ 质量密度，

$\varphi_i\to\varphi(x,t)$ 连续空间与时间的函数. field 场!

光滑 $\Rightarrow \frac{\varphi_{i+1}-\varphi_i}{a}=\frac{\varphi(x+dx)-\varphi(x)}{dx}\longrightarrow\frac{\partial\varphi(x)}{\partial x}$，$ka\longrightarrow\rho=$ Young's modulus，$\frac{\rho}{2}\left(\frac{\partial\varphi}{\partial x}\right)^2$：单位长度势能

$$L=\int\mathcal{L}\,dx,\quad \text{其中}\quad \boxed{\mathcal{L}=\frac{1}{2}\left[\sigma\dot{\varphi}^2-\rho\left(\frac{\partial\varphi}{\partial x}\right)^2\right]}\longleftarrow \text{拉氏密度}$$

$$\delta S=\delta\int_{t_i}^{t_f}L\,dt=\delta\int_{t_i}^{t_f}dt\int dx\,\mathcal{L}\left(\varphi,\dot{\varphi},\frac{\partial\varphi}{\partial x}\right)$$

$$=\int dt\int dx\left\{\frac{\partial\mathcal{L}}{\partial\varphi}\delta\varphi+\frac{\partial\mathcal{L}}{\partial(\partial\varphi/\partial x)}\delta\left(\frac{\partial\varphi}{\partial x}\right)+\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\varphi}{\partial t}\right)}\delta\left(\frac{\partial\varphi}{\partial t}\right)\right\}$$

$$=\int dt\int dx\left\{\frac{\partial\mathcal{L}}{\partial\varphi}\delta\varphi-\frac{\partial}{\partial x}\left(\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\varphi}{\partial x}\right)}\right)\delta\varphi-\frac{\partial}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\varphi}{\partial t}\right)}\right)\delta\varphi\right\}$$

这里作了分部积分，利用了在 t_i 与 t_f，x 边界，φ 值确定，即 $\delta\varphi=0$

$$\delta S=0\quad\Rightarrow\quad\boxed{\frac{\partial}{\partial x}\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\varphi}{\partial x}\right)}+\frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\varphi}{\partial t}\right)}-\frac{\partial\mathcal{L}}{\partial\varphi}=0}\quad\text{拉格朗日方程}$$

对于我们的弹簧链，就是

$$\boxed{\rho\frac{\partial^2\varphi}{\partial x^2}-\sigma\frac{\partial^2\varphi}{\partial t^2}=0}$$

这是波动方程，波速 $v=\sqrt{\rho/\sigma}$ $(=\omega/k)$ 解：$\varphi=Ae^{ikx-i\omega t}$

$\mathcal{L}$对应 Hamiltonian density：$\mathcal{L}=\dot{\varphi}\frac{\partial\mathcal{L}}{\partial\dot{\varphi}}-\mathcal{H}$，$\frac{\partial\mathcal{L}}{\partial\dot{\varphi}}$ 对应动量 $\pi(x,t)$

$$\boxed{\mathcal{H}=\frac{1}{2}\sigma\dot{\varphi}^2+\frac{1}{2}\rho\left(\frac{\partial\varphi}{\partial x}\right)^2=T+V}$$

推广到二维：mattress（席梦思）

$$\mathcal{L}(\varphi) = \frac{1}{2}\sigma\left(\frac{\partial\varphi}{\partial t}\right)^2 - \frac{1}{2}\rho\left[\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2\right] \qquad S = \int_{t_i}^{t_f} dt \int d^2x\, \mathcal{L}(\varphi)$$

$$\delta S = 0 \Rightarrow -\sigma\frac{\partial^2\varphi}{\partial t^2} + \rho\left(\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2}\right) = 0 \quad \text{即} \quad \boxed{\frac{\partial^2\varphi}{\partial t^2} - c^2\nabla^2\varphi = 0} \quad c = \sqrt{\rho/\sigma}\ \text{为声速}$$

更一般地，考虑质点之间互作用，$\mathcal{L}(\varphi)$中还包括：

$$-\tau\varphi^2 - \varsigma\varphi^4 + \cdots$$

定义 $x_0 = ct$, $x_1 = x$, $x_2 = y$, $\varphi \to \frac{\varphi}{\sqrt{\sigma}}$ 即 $dt\,\rho\left(\frac{\partial\varphi}{\partial x_1}\right)^2 \to c\,dx_0\left(\frac{\partial\varphi}{\partial x_1}\right)^2$, $dt\,\sigma\left(\frac{\partial\varphi}{\partial t}\right)^2 \to c\,dx_0\left(\frac{\partial\varphi}{\partial x_0}\right)^2$

$$\overset{c=1}{\Longrightarrow} S = \int d^dx\left[\frac{1}{2}(\partial\varphi)^2 + \frac{1}{2}m^2\varphi^2 - \frac{1}{4!}\lambda\varphi^4 + \cdots\right],$$ 其中：$(\partial\varphi)^2 = \left(\frac{\partial\varphi}{\partial x_0}\right)^2 - \left(\frac{\partial\varphi}{\partial x}\right)^2 - \left(\frac{\partial\varphi}{\partial y}\right)^2$，$m, \lambda$为参数

这就是 Landau-Ginzburg 作用量，$\mathcal{L} = \frac{1}{2}(\partial\varphi)^2 - V(\varphi)$

- 对称性决定其形式：$(\partial\varphi)^2$ 具有 Lorentz-invariant（c是光速），$\varphi \to -\varphi$不变 $\Rightarrow V(\varphi)$中只有φ的偶次

现代物理学：根据想要的对称性及想描述的场的直接规律构造作用量，这里是标量场 φ.

- 将我们对席梦思上一个质点的讨论推广到整个席梦思，我们就得到了

场 $\varphi(\vec{x}, t) = \varphi(x)$ 的量子场论：

$$Z = \int \mathcal{D}\varphi\, e^{\frac{i}{\hbar}S}, \qquad S = \int d^dx\, \mathcal{L}(\varphi)$$

- $\varphi(x)$ 是 $d+1$ 维时空（space-time）中场. 又如谐振子 $0+1$ 维场；振子链由 $1+1$ 维场，Mattress：$(2+1)$维

例：4维时空中 $\mathcal{L} = \frac{1}{2}\eta^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - \frac{1}{2}m^2\varphi^2 \qquad \eta = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$ 度规 metric

其中约定：$\partial_\mu\varphi\,\partial^\mu\varphi \equiv \eta^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi = (\partial\varphi)^2$，$\partial_\mu\partial^\mu\varphi = \eta^{\mu\nu}\partial_\nu\partial_\mu\varphi$

$$\delta S = 0 \Rightarrow -\frac{\partial\mathcal{L}}{\partial\varphi} + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\right) = 0 \Rightarrow \boxed{\ddot{\varphi} - \nabla\cdot\nabla\varphi + m^2\varphi = 0}$$

简写成 $\boxed{\partial_\mu\partial^\mu\varphi + m^2\varphi = 0}$ “经典” Klein-Gordon 方程.

约定：1. 重复指标求和 2. $a^\mu = (a^0, \vec{a})$ 反协变4矢，$a_\mu = \eta_{\mu\nu}a^\nu = (a^0, -\vec{a})$
$a^\mu = \eta^{\mu\nu}a_\nu$