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Exact finite-size corrections and corner free energies for the $c = -2$ universality class

Nickolay Izmailian [a,b,*], Ralph Kenna [a], Wenan Guo [c], Xintian Wu [c]

[a] *Applied Mathematics Research Center, Coventry University, Coventry CV1 5FB, UK*
[b] *Yerevan Physics Institute, Alikhanian Brothers 2, 375036 Yerevan, Armenia*
[c] *Department of Physics, Beijing Normal University, Beijing 100875, China*

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Abstract

We consider the partition functions of the anisotropic dimer model on the rectangular $(2M - 1) \times (2N - 1)$ lattice with (a) free and (b) cylindrical boundary conditions with a single monomer residing on the boundary. We express (a) and (b) in terms of a principal partition function with twisted boundary conditions. Based on these expressions, we derive the exact asymptotic expansions of the free energy for both cases (a) and (b). We confirm the conformal field theory prediction for the corner free energy of these models, and find the central charge is $c = -2$. We also show that the dimer model on the cylinder with an odd number of sites on the perimeter exhibits the same finite-size corrections as on the plane.

1. Introduction

Finite-size scaling has been of interest to scientists working on a variety of critical systems, including spin models, percolation models, lattice gauge models, spin glass, etc. [1]. The properties of the associated corrections to theoretical predictions for the behavior of idealized infinite systems play increasingly important roles in improving understanding of real statistical systems

* Corresponding author.
E-mail addresses: izmail@yerphi.am, ab5223@coventry.ac.uk (N. Izmailian), r.kenna@coventry.ac.uk (R. Kenna), waguo@bnu.edu.cn (W. Guo), wuxt@bnu.edu.cn (X. Wu).

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in the critical regime. Therefore, in recent decades there have been many investigations on finite-size scaling, finite-size corrections, and boundary effects for model systems.

To fully understand such effects, analyses which can be carried out without delivering numerical errors are of particular importance. These include systems such as the Ising model [2–4], spanning-tree models [5], dimer models [5–10], the critical dense polymer model [10,11], resistor network [12], Hamiltonian walk [13], critical Potts model [14], etc., which allow for exact studies.

Models with exact solutions, therefore, have key roles in understanding the forms of finite-size scaling. Ferdinand and Fisher stimulated such studies [2] by performing a finite-size analysis of Onsager's exact solution [15] of the two-dimensional Ising model on finite-size rectangular lattices with periodic boundary conditions. Although the solution for the Ising model with free boundaries is still lacking, exact solutions for a variety of different models with various boundaries have been obtained and studied intensively.

Many critical systems have been shown to have local scale invariance, so that their scaling limits can be described by conformal field theory. Such a theory is parameterized by the value of its central charge c, which itself is related to the finite-size corrections to the critical free energy. For critical two-dimensional systems, it has long been known that the free energy contains a term of order $O(\ln L)$ due to corner singularities, where L is the typical size of the system, with a universal prefactor proportional to the central charge [16]. The study of statistical systems in the presence of such corner singularities has emerged as a topic in its own right – one which is increasingly gained in importance [16–21].

Several years ago an efficient bond-propagation algorithm was developed for computing the partition function of the Ising model with free edges and corners in two dimensions [22]. With this algorithm, calculations have been carried out on various lattices and the results are accurate to a remarkable margin of 10^{-26} [19–21]. Fitting the standard finite-size scaling formulae to associated data allowed the edge and corner terms for the free energy to be obtained very accurately. For example, from the corner term for the rectangular lattice (comprising rectangular elementary plaquette) [19] and from the corner terms for triangular, rhomboid, trapezoid, hexagonal and rectangular lattices (each comprising elementary triangular plaquette) [20], the central charge of the Ising model was estimated to be $c = 0.5 \pm 1 \times 10^{-10}$, compared with the conformal field theory result $c = 0.5$ [16].

Conformal invariance implies that on a finite lattice with free boundaries the critical free energy has the generic form [17]

$$F = \mathcal{S} f_{\text{bulk}} + \mathcal{P} f_{\text{surf}} + f_0 + O(1/\mathcal{S}). \tag{1}$$

Here $\mathcal{S}$ represents the area of the lattice and f_{bulk} is the free energy per unit area. The second term represents contributions from the lattice perimeter $\mathcal{P}$, with f_{surf} the associated free energy per unit edge length.

In general, the coefficients f_{bulk} and f_{surf} are non-universal, but the coefficient f_0 is supposed [23] to be universal, depending only on the shape of the system and, possibly, the nature of the boundary conditions. In some two-dimensional geometries, the value of f_0 is known [24] to be simply related to the conformal anomaly number c of the theory. Cardy and Peschel [16] have shown that corners on the boundary induce a trace anomaly in the stress tensor. This gives rise to a term in f_0 proportional to $\ln \mathcal{S}$, where $\mathcal{S}$ is the area of the domain. Later, Kleban and Vassileva [17] have shown that in rectangular geometry in addition to corner contribution proportional to $\ln \mathcal{S}$ the term f_0 contains a term depending on the aspect ratio, namely, the term f_0 contains the universal part f_{univ} given by

$$f_{\text{univ}} = -\frac{c}{8}\ln\mathcal{S} + \frac{c}{4}\ln\left[\eta(q)\eta(q')\right]. \tag{2}$$

Here c is the central charge, $q = \exp(-2\pi\xi)$, $q' = \exp(-2\pi/\xi)$, ξ is the aspect ratio and η the Dedekind eta function. However, Kleban and Vassileva mentioned that, in their derivation, a possible non-universal additive constant was not included [17].

In this paper we derive exact asymptotic expansions, to arbitrary order, for the free energies of critical systems described by logarithmic conformal field theory with central charge $c = -2$. Such systems are realized as the dimer model on a rectangular lattice, the Abelian sandpile model, the spanning tree, Hamiltonian walks on a Manhattan lattice, rational triplet theory, symplectic fermions, the traveling salesman problem, as well as branching polymers. The calculation of the central charge, based on finite-size corrections for the dimer model on the rectangular lattice, has led to some confusion in the literature, due to the (mis)interpretation of finite-size corrections in terms of the central charge rather than the effective central charge [5,25]. Only quite recently it has been shown [8–10] that the central charge for the dimer model is $c = -2$. In particular, we consider the anisotropic dimer model on a $(2M-1)\times(2N-1)$ rectangular lattice with (a) free and (b) cylindrical boundary conditions with a single monomer on the boundary. We show that the exact asymptotic expansion for the free energy for these models can be written as

$$F = \mathcal{S} f_{\text{bulk}} + 2\mathcal{N} f_{1s}(x,y) + 2\mathcal{M} f_{2s}(x,y) + f_0(z\xi) + \sum_{p=1}^{\infty} \frac{f_p(z\xi)}{\mathcal{S}^p}, \tag{3}$$

where $\mathcal{S} = \mathcal{M}\mathcal{N}$, f_{1s} and f_{2s} are the free energies per unit edge length in the horizontal and vertical directions respectively, along which x and y are the dimer weights with $z = x/y$. The quantities $\mathcal{M}$ and $\mathcal{N}$ are functions of the physical dimensions of the lattice and the aspect ratio is $\xi = \mathcal{N}/\mathcal{M}$. *All* coefficients in the expansion (3) are expressed through analytical functions.

The correspondences between $\mathcal{M}$ and $\mathcal{N}$ in Eq. (3) and the lattice dimensions for system (a) and (b) are summarized as:

$$(\mathcal{M},\mathcal{N}) = \begin{cases} (2M, 2N) & \text{for the dimer model with free boundary conditions,} \\ (2M-1, 2N) & \text{for the dimer model on cylinder.} \end{cases}$$

We show that f_0 contains the universal part f_{univ} given by Eq. (2). This confirms the conformal field theory prediction for the corner free energy in models for which the central charge is $c = -2$.

The paper is organized as follows. In Section 2 we show how to express the partition function for the dimer model on a $(2M-1)\times(2N-1)$ rectangular lattice with (a) free and (b) cylindrical boundary conditions with a single monomer on the boundary in the form of a partition function with twisted boundary conditions. In Section 3 asymptotic expansions of the free energies are presented. The results are summarized and discussed in Section 4.

2. Partition function of the dimer model

Consider a anisotropic dimer model on a finite rectangular lattice with an odd number of rows and an odd number of columns. The lattice is planar if there are free boundary conditions in both directions. It is cylindrical if there are periodic boundary conditions in the horizontal direction, for example, and free boundary conditions in the vertical direction. The partition function for the anisotropic dimer model is given by

$$Z(x,y) = \sum x^{n_h} y^{n_v}, \tag{4}$$

where the summation is taken over all dimer covering configurations. Here, n_h and n_v are the number of horizontal and vertical dimers whose weights are x and y respectively. It has been shown that the exact partition functions of the anisotropic dimer model on finite rectangular lattices with free, cylindrical, toroidal, Möbius-strip and Klein-bottle boundary conditions can be expressed in terms of the principal object

$$Z^2_{\alpha,\beta}(z, M, N) = \prod_{n=0}^{N-1} \prod_{m=0}^{M-1} 4\left[z^2 \sin^2 \frac{(m+\alpha)\pi}{M} + \sin^2 \frac{(n+\beta)\pi}{N}\right], \tag{5}$$

where $(\alpha, \beta) = (1/2, 0), (0, 1/2)$ or $(1/2, 1/2)$ and $z = x/y$. Here, M and N are related to the lattice dimensions, the precise details depending on the lattice geometry in question [7].

It is clear that $Z_{0,0}(z, M, N)$ vanishes due to the zero mode at $(m, n) = (0, 0)$. In what follows, therefore, we remove the zero mode, and when $\alpha = \beta = 0$ replace $Z_{0,0}(z, M, N)$ in Eq. (5) by

$$Z^2_{0,0}(z, M, N) = \prod_{n=0}^{N-1} {\prod_{m=0}^{M-1}}' 4\left[z^2 \sin^2 \frac{m\pi}{M} + \sin^2 \frac{n\pi}{N}\right], \tag{6}$$

where the prime on the product denotes the restriction $(m, n) \neq (0, 0)$.

The general theory for the asymptotic expansion of $Z_{\alpha,\beta}(z, M, N)$ for $(\alpha, \beta) \neq (0, 0)$ appearing in the anisotropic dimer model has been given in [7]. In this paper we will present the asymptotic expansion of $Z_{0,0}(z, M, N)$.

In this section, we consider the anisotropic dimer model on $(2M - 1) \times (2N - 1)$ rectangular lattices with (a) free and (b) cylindrical boundary conditions with a single monomer on the boundary. The aim of the section is to show that the partition functions for both (a) and (b) can be written in terms of the principal mathematical object appearing in Eq. (6).

2.1. Dimer model on the rectangular lattice with free boundary condition

Let us first consider the anisotropic dimer model on $(2M - 1) \times (2N - 1)$ rectangular lattices with free boundary conditions with a single monomer on the boundary. The lattice is bipartite and consists of two sublattice A and B. Since the total number of sites is odd the four corner sites belong to the same sublattice, say, A and there are one more A than B sites. The lattice can therefore be completely covered by dimers if we put monomer on the boundary to one of the site belong to the sublattice A.

The exact partition function for the dimer model on a $(2M - 1) \times (2N - 1)$ rectangular lattice with free boundary conditions and with a single monomer on the boundary is given by [5,26]

$$\begin{aligned} Z^{\text{free}}_{2M-1,2N-1} &= x^{M-1} y^{N-1} \prod_{n=1}^{N-1} \prod_{m=1}^{M-1} 4\left[x^2 \cos^2 \frac{m\pi}{2M} + y^2 \cos^2 \frac{n\pi}{2N}\right] \\ &= z^{M-1} y^{2MN-M-N} \prod_{n=1}^{N-1} \prod_{m=1}^{M-1} 4\left[z^2 \cos^2 \frac{m\pi}{2M} + \cos^2 \frac{n\pi}{2N}\right]. \end{aligned} \tag{7}$$

This result is independent of the location of the monomer provided that it is an A site on the boundary. We change the variables $n \to N - n$ and $m \to M - m$ to write the partition function as

$$Z^{\text{free}}_{2M-1,2N-1} = z^{M-1} y^{2MN-M-N} \prod_{n=1}^{N-1} \prod_{m=1}^{M-1} 4\left[z^2 \sin^2 \frac{m\pi}{2M} + \sin^2 \frac{n\pi}{2N}\right]. \tag{8}$$

It is easy to show that $f(2N-n,m) = f(n,2M-m) = f(n,m)$, where

$$f(n,m) = 4\left[z^2 \sin^2\frac{m\pi}{2M} + \sin^2\frac{n\pi}{2N}\right]. \tag{9}$$

This allows us to express the double product $\prod_{n=0}^{2N-1}\prod_{m=0}^{\prime 2M-1} f(n,m)$ in terms of the simpler $\prod_{n=1}^{N-1}\prod_{m=1}^{M-1} f(n,m)$ through

$$\begin{aligned}&\prod_{n=0}^{2N-1}\prod_{m=0}^{2M-1}{}' f(n,m)\\&\quad= \frac{\prod_{n=1}^{2N-1} f(n,M)f(n,0)\prod_{m=1}^{2M-1} f(N,m)f(0,m)}{f(N,M)}\left[\prod_{n=1}^{N-1}\prod_{m=1}^{M-1} f(n,m)\right]^4,\end{aligned} \tag{10}$$

with $f(N,M) = 4(1+z^2)$. The prime on the product again denotes the restriction $(m,n) \neq (0,0)$. With the help of the identities [28]

$$4\sinh^2(M\omega) = \prod_{m=0}^{M-1} 4\left[\sinh^2\omega + \sin^2\frac{m\pi}{M}\right], \tag{11}$$

and

$$\prod_{m=1}^{M-1} 4\sin^2\frac{m\pi}{M} = M^2, \tag{12}$$

the products $\prod_{n=1}^{2N-1} f(n,M)f(n,0)$ and $\prod_{m=1}^{2M-1} f(N,m)f(0,m)$ can be written as

$$\prod_{n=1}^{2N-1} f(n,M)f(n,0) = \frac{4N^2}{z^2}\sinh^2(2N \operatorname{arcsinh} z), \tag{13}$$

and

$$\prod_{m=1}^{2M-1} f(N,m)f(0,m) = 4M^2 z^{8M-2}\sinh^2\left(2M \operatorname{arcsinh}\frac{1}{z}\right), \tag{14}$$

respectively. Now, using Eqs. (6), (8)–(10), (13) and (14) the partition function for dimers with free boundary conditions can finally be written as

$$Z_{2M-1,2N-1}^{\text{free}} = Q Z_{0,0}^{1/2}(z, 2M, 2N), \tag{15}$$

in which

$$Q = \frac{y^{2MN-M-N}}{z^M}\frac{(1+z^2)^{1/4}}{\sqrt{2MN\sinh(2N \operatorname{arcsinh} z)\sinh(2M \operatorname{arcsinh} 1/z)}}. \tag{16}$$

2.2. Dimer model on the rectangular lattice with cylindrical boundary conditions

Now, let us consider the anisotropic dimer model on $(2M-1)\times(2N-1)$ rectangular lattices with cylindrical boundary conditions with a single monomer on the boundary. The lattice is nonbipartite and cannot be divided to the two sublattice A and B. The lattice can be fully covered by one monomer and dimers. In 1974 Temperley [27] introduce a bijection between dimer configurations with single monomer on the boundary of a planar lattice and spanning trees on a related lattice. However, the success of the Temperley bijection apparently depends on the lattice being bipartite; it does not work for nonbipartite lattices. By using an alternate mapping [26] the anisotropic dimer model on $(2M-1)\times(2N-1)$ rectangular lattices with cylindrical boundary conditions with a single monomer on the boundary has been solved in [29].

The exact partition function for the dimer model on a $(2M-1)\times(2N-1)$ rectangular lattice, with cylindrical boundary conditions, with a single monomer on the boundary is given by [29]

$$\begin{aligned} Z^{\mathrm{cyl}}_{2M-1,2N-1} &= x^{M-1}y^{N-1}\prod_{n=1}^{N-1}\prod_{m=1}^{M-1}4\left[x^2\sin^2\frac{2m\pi}{2M-1}+y^2\cos^2\frac{n\pi}{2N}\right] \\ &= z^{M-1}y^{2MN-M-N}\prod_{n=1}^{N-1}\prod_{m=1}^{M-1}4\left[z^2\sin^2\frac{2m\pi}{2M-1}+\cos^2\frac{n\pi}{2N}\right]. \end{aligned} \tag{17}$$

Using the transformation $n\to 2N-n$ and the relation

$$\prod_{m=1}^{M-1}\left(a+\sin^2\frac{2m\pi}{2M-1}\right)=\prod_{m=1}^{M-1}\left(a+\sin^2\frac{m\pi}{2M-1}\right), \tag{18}$$

the partition function given by Eq. (17) can be expressed in the form

$$Z^{\mathrm{cyl}}_{2M-1,2N-1} = z^{M-1}y^{2MN-M-N}\prod_{n=1}^{N-1}\prod_{m=1}^{M-1}4\left[z^2\sin^2\frac{m\pi}{2M-1}+\sin^2\frac{n\pi}{2N}\right]. \tag{19}$$

Following the same procedure as in the case of free boundary conditions, we obtain

$$Z^{\mathrm{cyl}}_{2M-1,2N-1} = RZ^{1/2}_{0,0}(z,2M-1,2N), \tag{20}$$

with

$$R=\frac{y^{2MN-M-N}}{z^{M-1/2}\sqrt{2N(2M-1)\sinh[(2M-1)\operatorname{arcsinh}1/z]}}. \tag{21}$$

Eqs. (15) and (20) give how the partition functions of the dimer model on a $(2M-1)\times(2N-1)$ rectangular lattice with (a) free and (b) cylindrical boundary conditions, with a single monomer on the boundary, can be expressed in terms of the principal object $Z_{0,0}(z,M,N)$. Based on such results, one can use the method proposed by Ivashkevich, Izmailian, and Hu [4] to derive the asymptotic expansion of the $Z_{0,0}(z,M,N)$ in terms of Kronecke'r double series [30], which are directly related to elliptic θ functions (see Appendix A).

3. Asymptotic expansion of free energy

In Section 2, we have shown that the partition functions of the dimer model with free and cylindrical boundary conditions with a single monomer on the boundary can be expressed in terms of the principal partition function with twisted boundary conditions $Z_{0,0}(z,M,N)$ (see Eqs. (15) and (20)). The asymptotic expansion of the $Z_{0,0}(z,M,N)$ is given in Appendix A.

After reaching this point, one can use Eq. (A.14) to write down all the terms of the exact asymptotic expansions of the free energy, $F = -\ln Z$ for all models under consideration in the form of Eq. (3).

The bulk free energy f_{bulk} for the dimer model on a $(2M-1)\times(2N-1)$ rectangular lattice for both free and cylindrical boundary conditions is given by

$$\begin{aligned} f_{\text{bulk}} &= -\frac{1}{2}\ln y - \frac{1}{2\pi}\int_0^\pi \omega_z(x)\,\mathrm{d}x \\ &= -\frac{1}{2}\ln y - \frac{1}{2\pi}\int_0^\pi \operatorname{arcsinh}(z\sin x)\,\mathrm{d}x \\ &= -\frac{1}{2}\ln y - \frac{\Phi(-z^2,2,\frac{1}{2})}{4\pi}, \end{aligned} \tag{22}$$

where $\omega_z(x)$ is lattice dispersion relation defined in Eq. (A.3) and $\Phi(x,s,\alpha)$ is the Lerch transcendent defined in Eq. (A.7). In particular for the isotropic dimer model ($z=1$), $\Phi(-1,2,1/2) = 4G$, where G is the Catalan constant given in Eq. (A.8) as $G = 0.915965594\ldots$. The surface free energy f_{1s} and f_{2s} defined by Eq. (3) are

$$f_{1s}^{\text{free}} = \frac{1}{4}\ln y + \frac{1}{4}\ln\bigl(z+\sqrt{1+z^2}\bigr), \tag{23}$$

$$f_{1s}^{\text{cyl}} = 0, \tag{24}$$

$$f_{2s}^{\text{free}} = f_{2s}^{\text{cyl}} = \frac{1}{4}\ln y + \frac{1}{4}\ln\bigl(1+\sqrt{1+z^2}\bigr). \tag{25}$$

For the leading correction terms $f_0(z\xi)$ we obtain

$$\begin{aligned} f_0^{\text{free}}(z\xi) &= \frac{1}{4}\ln S - \frac{1}{4}\ln\xi - \ln\eta(iz\xi) - \frac{3}{2}\ln 2 - \frac{1}{4}\ln\left(1+z^2\right) \\ &= \frac{1}{4}\ln S - \frac{1}{2}\ln\eta(iz\xi)\eta\bigl(i/(z\xi)\bigr) - \frac{3}{2}\ln 2 - \frac{1}{4}\ln\left(1+z^2\right) + \frac{1}{4}\ln z, \end{aligned} \tag{26}$$

in which $\xi = \frac{N}{M}$,

$$\begin{aligned} f_0^{\text{cyl}}(z\xi) &= \frac{1}{4}\ln S - \frac{1}{4}\ln\xi - \ln\eta(iz\xi) - \frac{1}{2}\ln 2 + \frac{1}{2}\ln y \\ &= \frac{1}{4}\ln S - \frac{1}{2}\ln\eta(iz\xi)\eta\bigl(i/(z\xi)\bigr) - \frac{1}{2}\ln 2 + \frac{1}{2}\ln y + \frac{1}{4}\ln z, \end{aligned} \tag{27}$$

in which $\xi = \frac{2N}{2M-1}$. Here η is the Dedekind eta function and we use the behavior of the this function under the Jacobi transformation $\tau \to \tau' = -1/\tau$,

$$\eta\bigl(\tau'\bigr) = \sqrt{-i\tau}\,\eta(\tau), \tag{28}$$

for $\tau = iz\xi$.

For the subleading correction terms $f_p(z\xi)$ for $p = 1,2,3,\ldots$, we obtain

$$f_p(z\xi) = \pi^{2p+1}\xi^{p+1}\frac{\Lambda_{2p}}{(2p)!}\frac{K_{2p+2}^{0,0}(iz\xi)}{2p+2},$$

where for free boundary conditions, we again use $\xi = \frac{N}{M}$ and for cylindrical boundary conditions $\xi = \frac{2N}{2M-1}$. As an example, we list first few expansion coefficients $f_p(z\xi)$ for $p = 1, 2$

$$
\begin{aligned}
f_1(z\xi) &= z\left(1+z^2\right)\frac{\pi^3\xi^2}{1080}\left(\theta_2^4\theta_3^4+\theta_3^4\theta_4^4-\theta_2^4\theta_4^4\right),\\
f_2(z\xi)&\\
&= \frac{\pi^5\xi^3}{12\,096}\left[\frac{z(1+z^2)(1+9z^2)}{5}+\frac{z^2(1+z^2)^2}{3}\frac{\partial}{\partial z}\right]\left(\theta_2^4+\theta_3^4\right)\left(\theta_2^4-\theta_4^4\right)\left(\theta_3^4+\theta_4^4\right), \qquad (29)
\end{aligned}
$$

where $\theta_i = \theta_i(z\xi)$ with $i = 2, 3, 4$.

Note, that with the help of the identities

$$
\begin{aligned}
&\frac{\partial}{\partial\xi}\ln\theta_3 = \frac{\pi}{4}\theta_4^4 + \frac{\partial}{\partial\xi}\ln\theta_2, \quad \text{and} \quad \frac{\partial}{\partial\xi}\ln\theta_4 = \frac{\pi}{4}\theta_3^4 + \frac{\partial}{\partial\xi}\ln\theta_2,\\
&\frac{\partial}{\partial\xi}\ln\theta_2 = -\frac{1}{2}\theta_3^2 E, \quad \text{and} \quad \frac{\partial E}{\partial\xi} = \frac{\pi^2}{4}\theta_3^2\theta_4^4 - \frac{\pi}{2}\theta_4^4 E,
\end{aligned}
$$

one can express all derivatives of the elliptic functions in terms of the elliptic functions $\theta_2, \theta_3, \theta_4$ and the elliptic integral of the second kind E.

From Eqs. (26) and (27) we can see that universal part of the f_0 is given by Eq. (2) with central charge $c = -2$. This proves the conformal field theory prediction for the corner free energy and shows that the corner free energy, which is proportional to the central charge c, is indeed universal.

It is interesting to note that leading finite-size corrections f_0 for the dimer model on a $(2M - 1) \times (2N - 1)$ square lattice with single monomer on the boundary is similar for both free and cylindrical boundary conditions. This similarity is unusual in integrable models and requires explanation [8]. Consider the dimer model on a $(2M - 1) \times (2N - 1)$ square lattice $\mathcal{L}$ with a single monomer on the boundary and with periodic boundary condition in the horizontal direction and free boundary conditions in the vertical direction. The lattice $\mathcal{L}$ then forms a cylinder of perimeter $(2N - 1)$ and height $(2M - 1)$. Let us enumerate the sites of the lattice $\mathcal{L}$ as (m, n), where $m = 1, 2, \dots, 2N - 1$ and $m = 1, 2, \dots, 2M - 1$. There is a bijection between dimer coverings on $\mathcal{L}$ with one boundary site removed and spanning trees on the odd–odd sublattice $G \subset \mathcal{L}$ with sites labeled as $(2n - 1, 2m - 1)$ with $n = 1, 2, \dots, N$ and $m = 1, 2, \dots, M$.

Let us select the odd–odd sublattice G and put monomer on the boundary. It is easy to see that two columns of G will be neighbours in G and in $\mathcal{L}$ (connected by horizontal bonds). Therefore a dimer may cover zero, one or two sites of G. The dimers covering no site of G are completely fixed by the others and play no role. For the others, we do the following construction. If a dimer touches only one site of G, we draw an arrow directed along the dimer from that site to the nearest neighbouring site of G. However, for a dimer laid on two sites of G, the two arrows would point from either site to the other, ruining the spanning tree picture. It can nevertheless be restored in the following way. Instead of seeing the two arrows as pointing from one site to its neighbour, we say that they point towards roots inserted between the neighbour sites, thus replacing the arrows ●→——←● by ●→○ ○←●. This in effect amounts to opening the cylinder by removing the horizontal bonds of $\mathcal{L}$ which connect sites of G, unwrapping it into a strip, and to adding columns of roots on the left and on the right side of the strip. The new arrow configurations define spanning trees, rooted anywhere on the left and right boundaries. So dimer coverings on the original cylinder are mapped to spanning trees on a strip, with close horizontal boundaries, and open vertical boundaries. Therefore, although the dimer model is originally defined on a cylinder, it shows the finite-size corrections expected on a strip, and must really be viewed as a model on a strip.

For odd–even $(2M-1)\times 2N$ and even–even $2M\times 2N$ cylinders with perimeter $2N$, the problem of having two arrows pointing from and to neighbor sites does not arise; however, the arrows one obtains do not define spanning trees but rather a combination of loops wrapped around the cylinder and tree branches attached to the loops.

4. Conclusion

We have derived the exact finite-size corrections for the free energy of the anisotropic dimer model on the $(2M-1)\times(2N-1)$ rectangular lattice with free and cylindrical boundary conditions with a single monomer on the boundary. We found that the exact asymptotic expansion of the free energy of the dimer model can be written in the form given by Eq. (3). We also show that the dimer model on the cylinder with an odd number of sites on the perimeter exhibits the same finite-size corrections as on the plane.

We proved the conformal field theory prediction about the corner free energy and have shown that the corner free energy, which is proportional to the central charge c, is indeed universal. We find the central charge in the framework of the conformal field theory to be $c=-2$.

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Appendix A. Asymptotic expansion of $Z_{0,0}(z,M,N)$

In this section we shall obtain the exact asymptotic expansion of the logarithm of the $Z_{0,0}(z,M,N)$. The logarithm of the $Z_{0,0}(z,M,N)$ can be expanded in the similar way as it has been done in Ref. [4] for the isotropic dimer model ($z_h=z_v$ or $z=1$).

$Z_{0,0}(z,M,N)$ can be transform in the following way

$$\begin{aligned} Z^2_{0,0}(z,M,N) &= \prod_{n=0}^{N-1}\prod_{m=0}^{M-1}{}' 4\left[z^2\sin^2\frac{m\pi}{M}+\sin^2\frac{n\pi}{N}\right] \\ &= \left(\prod_{n=1}^{N-1}4\sin^2\frac{n\pi}{N}\right)\prod_{n=0}^{N-1}\prod_{m=1}^{M-1}4\left[z^2\sin^2\frac{m\pi}{M}+\sin^2\frac{n\pi}{N}\right] \\ &= N^2\prod_{n=0}^{N-1}\prod_{m=1}^{M-1}4\left[z^2\sin^2\frac{m\pi}{M}+\sin^2\frac{n\pi}{N}\right]. \end{aligned} \tag{A.1}$$

With the help of the identity given by Eq. (11), $Z_{0,0}(z,M,N)$ can be transformed into a simpler form

$$Z_{0,0}(z, M, N) = N \prod_{m=1}^{M-1} 2\sinh\left[N\omega_z\left(\frac{m\pi}{M}\right)\right] \tag{A.2}$$

where lattice dispersion relation is

$$\omega_z(k) = \operatorname{arcsinh}(z \sin k). \tag{A.3}$$

Considering the logarithm of $Z_{0,0}(z, M, N)$, we note, that it can be transformed as

$$\ln Z_{0,0}(z, M, N) = \ln N + N \sum_{m=1}^{M-1} \omega_z\left(\frac{\pi m}{M}\right) + \sum_{m=1}^{M-1} \ln\left[1 - e^{-2N\omega_z(\frac{\pi m}{M})}\right]. \tag{A.4}$$

The second sum here vanishes in the formal limit $N \to \infty$ when the system turns into infinitely long strip of width M. The asymptotic expansion of the first sum can be found with the help of the Euler–Maclaurin summation formula [31]

$$N \sum_{m=0}^{M-1} \omega_z\left(\frac{\pi m}{M}\right) = \frac{S}{\pi} \int_0^\pi \omega_z(x)\mathrm{d}x - \pi z \xi B_2 - 2\pi\xi \sum_{p=1}^{\infty} \left(\frac{\pi^2 \xi}{S}\right)^p \frac{z_{2p}}{(2p)!} \frac{B_{2p+2}}{2p+2}, \tag{A.5}$$

where B_p are so-called Bernoulli numbers ($B_2 = 1/6$, $B_4 = -1/30$, $B_6 = 1/42, \ldots$) and

$$\int_0^\pi !\omega_z(x)\mathrm{d}x = \frac{1}{2}\Phi\left(-z^2, 2, \frac{1}{2}\right), \tag{A.6}$$

where $\Phi(x, s, \alpha)$ is the Lerch transcendent defined as

$$\Phi(x, s, \alpha) = \sum_{n=0}^{\infty} (\alpha + n)^{-s} x^n. \tag{A.7}$$

In particular, for isotropic dimer model ($z = 1$), the Lerch transcendent is now $\Phi(-1, 2, 1/2) = 4G$, where G is the Catalan constant given by

$$G = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} = 0.915965594\ldots. \tag{A.8}$$

We have also used the symmetry property, $\omega_z(k) = \omega_z(\pi - k)$, of the lattice dispersion relation (A.3) and its Taylor expansion

$$\omega_z(k) = k\left(z + \sum_{p=1}^{\infty} \frac{z_{2p}}{(2p)!} k^{2p}\right), \tag{A.9}$$

with $z_2 = -z(1 + z^2)/3$, $z_4 = z(1 + z^2)(1 + 9z^2)/5$, $z_6 = -z(1 + z^2)(1 + 90z^2 + 225z^4)/7$, etc.

The second sum in Eq. (A.4) can be analyzed in the following way: we first expand $\ln(1 - e^A)$ as a power series in e^A and then split the sum in two part: $m \in [0, [M/2] - 1]$ and $m \in [[M/2], M - 1]$ and finally we change variable m in the second part viz. $m \to M - m$. As result we obtain

$$\sum_{m=1}^{M-1} \ln\left[1-e^{-2N\omega_z(\frac{\pi m}{M})}\right] = -\sum_{n=1}^{\infty}\frac{1}{n}\left\{\sum_{m=1}^{[M/2]-1} e^{-2nN\omega_z(\frac{\pi m}{M})]} + \sum_{m=1}^{M-[M/2]} e^{-2nN\omega_z(\frac{\pi m}{M})]}\right\}. \tag{A.10}$$

The argument of the exponent can be expanded in powers of $1/S$ if we replace the lattice dispersion relation $\omega_z(x)$ with its Taylor expansion (A.9)

$$\exp\left[-2nN\omega_z\left(\frac{\pi m}{M}\right)\right] = \exp\left\{-2\pi mnz\xi - 2\pi n\xi\sum_{p=1}^{\infty}\frac{z_{2p}}{(2p)!}\left(\frac{\pi^2\xi}{S}\right)^p m^{2p+1}\right\},$$

where $\xi = M/N$. Taking into account the relation between moments and cumulants (see Appendix B), we obtain asymptotic expansion of the first exponent itself in powers of $1/S$

$$e^{-2nN\omega_z(\frac{\pi m}{M})} = e^{-2\pi nmz\xi} - 2\pi n\xi\sum_{p=1}^{\infty}\left(\frac{\pi^2\xi}{S}\right)^p\frac{\Lambda_{2p}}{(2p)!}m^{2p+1}e^{-2\pi nmz\xi}.$$

The differential operators Λ_{2p} that have appeared here can be expressed via coefficients z_{2p} of the expansion of the lattice dispersion relation as

$$\Lambda_2 = z_2,$$
$$\Lambda_4 = z_4 + 3z_2^2\frac{\partial}{\partial z},$$
$$\Lambda_6 = z_6 + 15z_4z_2\frac{\partial}{\partial} + 15z_2^3\frac{\partial^2}{\partial z^2}.$$
$$\vdots$$

Plugging the expansion of the exponent back into Eq. (A.10) we obtain

$$\sum_{m=1}^{M-1} \ln\left[1-e^{-2N\omega_z(\frac{\pi m}{M})}\right] = -\sum_{n=1}^{\infty}\frac{1}{n}\left\{\sum_{m=1}^{[M/2]-1} e^{-2\pi nmz\xi} + \sum_{m=1}^{M-[M/2]} e^{-2\pi nmz\xi}\right\} + 2\pi\xi\sum_{p=1}^{\infty}\left(\frac{\pi^2\xi}{S}\right)^p\frac{\Lambda_{2p}}{(2p)!}\sum_{n=1}^{\infty}\left\{\sum_{m=1}^{[M/2]-1} m^{2p+1}e^{-2\pi nmz\xi} + \sum_{m=1}^{M-[M/2]} m^{2p+1}e^{-2\pi nmz\xi}\right\}.$$

In all these series, summation over m can be extended to infinity. The resulting errors are exponentially small and do not affect our asymptotic expansion in any finite power of $1/S$. As result we obtain

$$\sum_{m=1}^{M-1} \ln\left[1-e^{-2N\omega_z(\frac{\pi m}{M})}\right] = -2\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{n}e^{-2\pi nmz\xi} + 4\pi\xi\sum_{p=1}^{\infty}\left(\frac{\pi^2\xi}{S}\right)^p\frac{\Lambda_{2p}}{(2p)!}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}m^{2p+1}e^{-2\pi nmz\xi}.$$

The key point of our analysis is the observation that all the series that have appeared in such an expansion can be obtained by resummation of either the Dedekind eta function, $\eta(\tau)$, or Kronecker's double series, $K_p^{0,0}(\tau)$.

The Dedekind eta function is usually defined as

$$\eta(\tau) = e^{\pi i \tau/12} \prod_{n=1}^{\infty} \left[1 - e^{2\pi i \tau n}\right].$$

Considering the logarithm of $\eta(\tau)$ of pure imaginary aspect ratio, $\tau = i\xi$, we obtain the identity

$$\ln \eta(i\xi) + \frac{\pi\xi}{12} = -\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{m} e^{-2\pi m n \xi}. \tag{A.11}$$

Kronecker's double series can be defined as [30]

$$K_p^{0,0}(\tau) = -\frac{p!}{(-2\pi i)^p} \sum_{m,n \in Z}{}' \frac{1}{(n + \tau m)^p},$$

where the prime over product denotes the restriction $(m, n) \neq (0, 0)$. In this form, however, they cannot be directly applied to our analysis. We need to cast them in a different form. The final result of our resummation of double Kronecker sum is

$$K_p^{0,0}(\tau) = B_p - p \sum_{m \neq 0} \sum_{n=0}^{\infty} n^{p-1} e^{2\pi i m n \tau}.$$

Considering the Kronecker sums with pure imaginary aspect ratio, $\tau = i\xi$, we can further rearrange this expression to get summation only over positive $m \geq 1$

$$B_{2p} - K_{2p}^{0,0}(i\xi) = 4p \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} n^{2p-1} e^{-2\pi m n \xi}. \tag{A.12}$$

Note, that Kronecker functions $K_{2p}^{0,0}(\tau)$ can all be expressed in terms of the elliptic θ-functions only (see Appendix C).

Now, with the help of the identities (A.11) and (A.12) we obtain

$$\sum_{m=0}^{M-1} \ln\left[1 - e^{-2N\omega_z(\frac{\pi m}{M})}\right] = 2\ln \eta(iz\xi) + \pi z \xi B_2 - 2\pi\xi \sum_{p=1}^{\infty} \left(\frac{\pi^2 \xi}{S}\right)^p \frac{\Lambda_{2p}}{(2p)!} \frac{K_{2p+2}^{0,0}(iz\xi) - B_{2p+2}}{2p+2}. \tag{A.13}$$

Substituting Eqs. (A.5) and (A.13) into Eq. (A.4) we finally obtain exact asymptotic expansion of the logarithm of $Z_{0,0}(z, M, N)$ in terms of the Kronecker's double series

$$\ln Z_{0,0}(z, M, N) = \frac{S}{\pi} \int_0^{\pi} \omega_z(x) \mathrm{d}x + \ln \sqrt{S\xi} + 2\ln \eta(iz\xi) - 2\pi\xi \sum_{p=1}^{\infty} \left(\frac{\pi^2 \xi}{S}\right)^p \frac{\Lambda_{2p}}{(2p)!} \frac{K_{2p+2}^{0,0}(iz\xi)}{2p+2}, \tag{A.14}$$

where $S = MN$, $\xi = N/M$. Note, that Bernoulli numbers B_p have finally dropped out from the asymptotic expansion.

Appendix B. Relation between moments and cumulants

In this appendix we consider the relation between moments Z_k and cumulants F_k which enters the expansion of exponent

$$\exp\left\{\sum_{k=1}^{\infty}\frac{x^k}{k!}F_k\right\} = 1 + \sum_{k=1}^{\infty}\frac{x^k}{k!}Z_k.$$

These are related in the following manner [32]

$$\begin{aligned}
Z_1 &= F_1,\\
Z_2 &= F_2 + F_1^2,\\
Z_3 &= F_3 + 3F_1F_2 + F_1^3,\\
Z_4 &= F_4 + 4F_1F_3 + 3F_2^2 + 6F_1^2F_2 + F_1^4,\\
&\ \ \vdots\\
Z_k &= \sum_{r=1}^{k}\sum\left(\frac{F_{k_1}}{k_1!}\right)^{i_1}\dots\left(\frac{F_{k_r}}{k_r!}\right)^{i_r}\frac{k!}{i_1!\dots i_r!},
\end{aligned}$$

where summation is over all positive numbers $\{i_1, \dots, i_r\}$ and different positive numbers $\{k_1, \dots, k_r\}$ such that $k_1 i_1 + \dots + k_r i_r = k$.

Appendix C. Reduction of Kronecker's double series to theta functions

The Laurent expansion of the Weierstrass function is given by

$$\begin{aligned}
\wp(z) &= \frac{1}{z^2} + \sum_{(n,m)\neq(0,0)}\left[\frac{1}{(z-n-\tau m)^2} - \frac{1}{(n+\tau m)^2}\right]\\
&= \frac{1}{z^2} + \sum_{p=2}^{\infty} a_p(\tau) z^{2p-2}.
\end{aligned}$$

The coefficients $a_p(\tau)$ of the expansion can all be written in terms of the elliptic θ-functions with the help of the recursion relation [33]

$$a_p = \frac{3}{(p-3)(2p+1)}(a_2a_{p-2} + a_3a_{p-3} + \dots + a_{p-2}a_2),$$

where first terms of the sequence are

$$\begin{aligned}
a_2 &= \frac{\pi^4}{15}\left(\theta_2^4\theta_3^4 - \theta_2^4\theta_4^4 + \theta_3^4\theta_4^4\right),\\
a_3 &= \frac{\pi^6}{189}\left(\theta_2^4 + \theta_3^4\right)\left(\theta_4^4 - \theta_2^4\right)\left(\theta_3^4 + \theta_4^4\right),\\
a_4 &= \frac{1}{3}a_2^2,
\end{aligned}$$

$$a_5 = \frac{3}{11}(a_2 a_3),$$
$$a_6 = \frac{1}{39}\left(2a_2^3 + 3a_3^2\right),$$
$$\vdots$$

Kronecker functions $K_{2p}^{0,0}(\tau)$ are related directly to the coefficients $a_p(\tau)$

$$K_{2p}^{0,0}(\tau) = -\frac{(2p)!}{(-4\pi^2)^p}\frac{a_p(\tau)}{(2p-1)}.$$

From the general formulas above we can easily write down all the Kronecker functions that have appeared in our asymptotic expansions in terms of the elliptic θ-functions, e.g.

$$K_4^{0,0}(\tau) = \frac{1}{30}\left(\theta_2^4\theta_4^4 - \theta_2^4\theta_3^4 - \theta_3^4\theta_4^4\right),$$
$$K_6^{0,0}(\tau) = \frac{1}{84}\left(\theta_2^4 + \theta_3^4\right)\left(\theta_4^4 - \theta_2^4\right)\left(\theta_3^4 + \theta_4^4\right),$$
$$K_8^{0,0}(\tau) = -\frac{1}{30}\left[\theta_2^4\theta_3^4 - \theta_2^4\theta_4^4 + \theta_3^4\theta_4^4\right]^2,$$
$$K_{10}^{0,0}(\tau) = \frac{5}{132}\left[\theta_2^4 + \theta_3^4\right]\left[\theta_4^4 + \theta_2^4\right]\left[\theta_3^4 + \theta_4^4\right]\left[\theta_2^4\theta_3^4 - \theta_2^4\theta_4^4 + \theta_3^4\theta_4^4\right],$$
$$\vdots$$

Note that when $\xi \to \infty$ we have limits $\theta_2 \to 0$, $\theta_4 \to 1$, $\theta_3 \to 1$ and the Kronecker's function reduce to the Bernoulli polynomials.

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