Quantum Magnetism and Quantum Criticality

- Classical and quantum phase transitions; RG and scaling
- Finite-size scaling of simulation data
- Example: dimerized Heisenberg models
- J-Q models; valence-bond solids without QMC sign problem
- Deconfined quantum criticality

Finite-size scaling - “phenomenological RG”

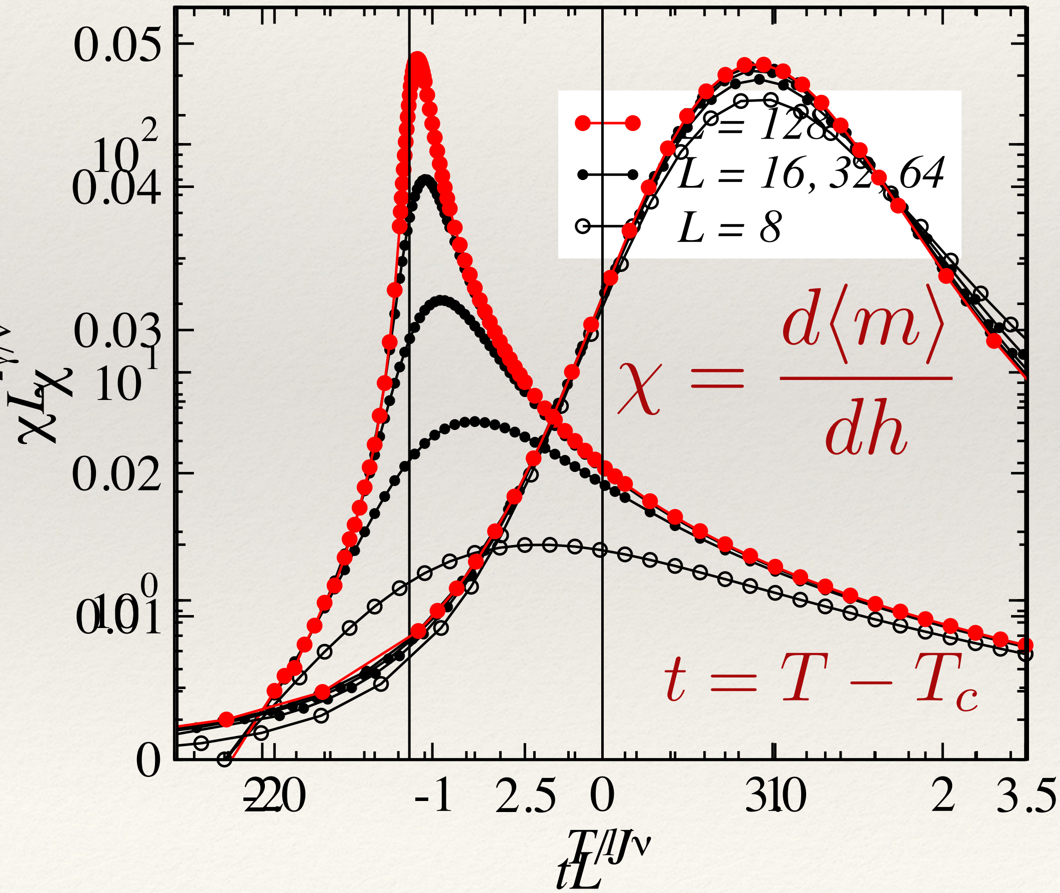
Energy scale lowered with increasing system size; L is length scale at criticality

Correlation length divergent for T → T_c $\xi \propto |\delta|^{-\nu}, \quad \delta = T - T_c$

Other singular quantity: $A(L \to \infty) \propto |\delta|^{\kappa} \propto \xi^{-\kappa/\nu}$

For **L-dependence** at T_c just let **ξ→L**: $A(T \approx T_c, L) \propto L^{-\kappa/\nu}$

Close to critical point: $A(L,T) = L^{-\kappa/\nu} g(\xi/L) = L^{-\kappa/\nu} f(\delta L^{1/\nu})$



2D Ising universality class

$\gamma = 7/4, \quad \nu = 1$

Critical T known

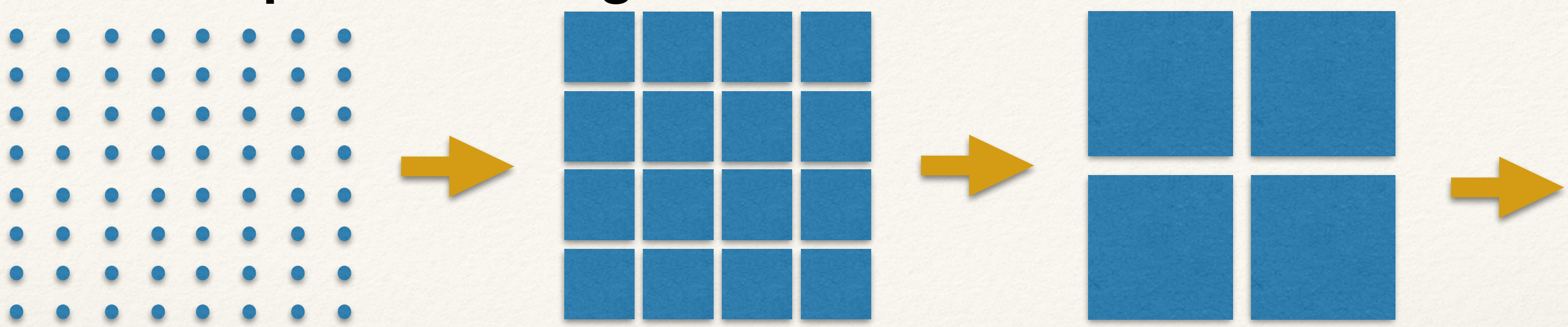
$T_c = 2/\ln(1+\sqrt{2}) \approx 2.2692$

When these are not known, treat as fitting parameters

- or extract in other way

Rough principles of renormalization

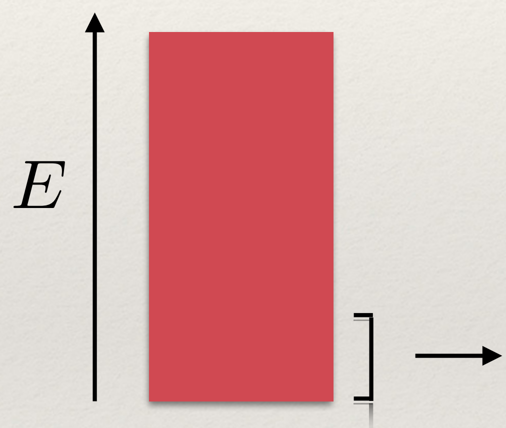
1) Kadanoff: Real-space blocking



Effective degrees of freedom of blocks reflect possible order parameters
- interactions between blocks evolve to some fixed point

2) Low-energy effective model

Repeated decimation of high-energy space
- interactions renormalize



renormalized interactions in low-energy subspace

…) many other schemes, practical or mainly conceptual

Essence: an infinite number of effective couplings λ_i
- flow as length scale is increased (energy scale is decreased): $\lambda_i(L) = \lambda_i^0 L_i^y$

Relevant (y_i>0) or irrelevant (yi<0), λ_i^0 depends on model parameters
- Very few relevant couplings ("fields")

Relevant and irrelevant perturbations of a critical point

Critical correlation function of some operator

$$\langle O(\vec{r}_1) O(\vec{r}_2) \rangle = \sum_i a_i r^{-2\Delta_i}$$

The scaling dimensions Δ_i correspond to the spectrum of 'orthogonal operators' (continuum fields) contained in the lattice operator *O*

- Loosely speaking, we say that the smallest Δ_i is the scaling dimension of *O*

Consider a **critical Hamiltonian H_0** and add some perturbation M

$$H = H_0 + h \textstyle\sum_i m_i = hM \ \ (\equiv hNm = hL^d m)$$

RG description of effects of hM at a critical point. Free energy density:

$$f_s(t, h, L) = L^{-d} F_s(tL^{1/\nu}, hL^y)$$

- t=0 at critical point; e.g., t=T-Tc (relevant field)

Taylor expand at t=0: $f_s^h \propto hL^{y-d}$

From Hamiltonian: $f_s^h = h\langle m \rangle \propto hL^{-\Delta}$

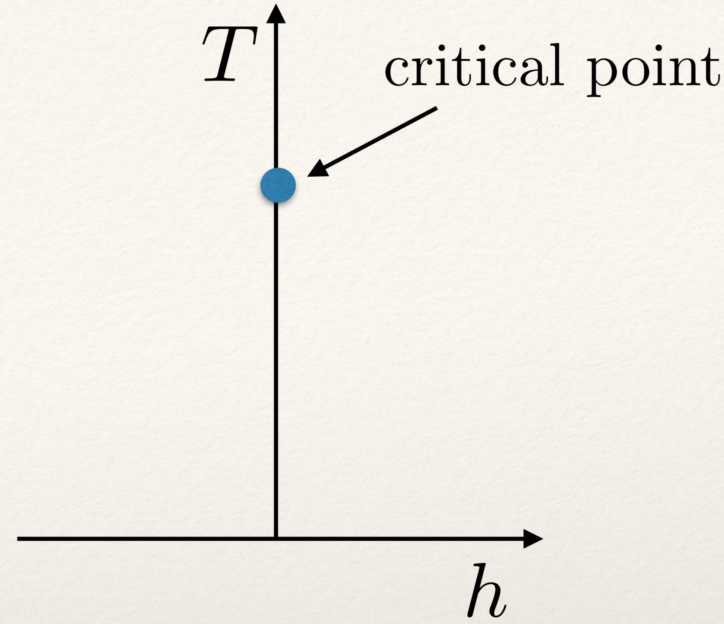
$$\rightarrow \ y = d - \Delta$$

y = scaling dimension of h

- The effect of the perturbation grows with L (it is relevant) only if y>0
- Irrelevant perturbation if y<0 (the critical point stays the same)
- A relevant perturbation changes the critical point in some way

Symmetric and symmetry-breaking fields

Example: Ising model
- classical model; energy and entropy



At h=0, T tunes to the critical point
- the ‘thermal field’ is $t=T-T_c$

Changing T changes the prefactor of E in

$$e^{-E(\sigma)/T}$$

- E is the operators conjugate to T

$$\langle E(r)E(0)\rangle \sim r^{-2\Delta_0}, \quad \Delta_0 = d - 1/\nu$$

Set t=0, tune the magnetic field; E → E+hM
- h ≠ 0 breaks the Z_2 symmetry of the model; relevant but not symmetric

$$\langle M(r)M(0)\rangle \sim r^{-2\Delta_M}, \quad \Delta_M = d - 1/\nu_M$$

The exponent Δ_M is related to the exponent we call η

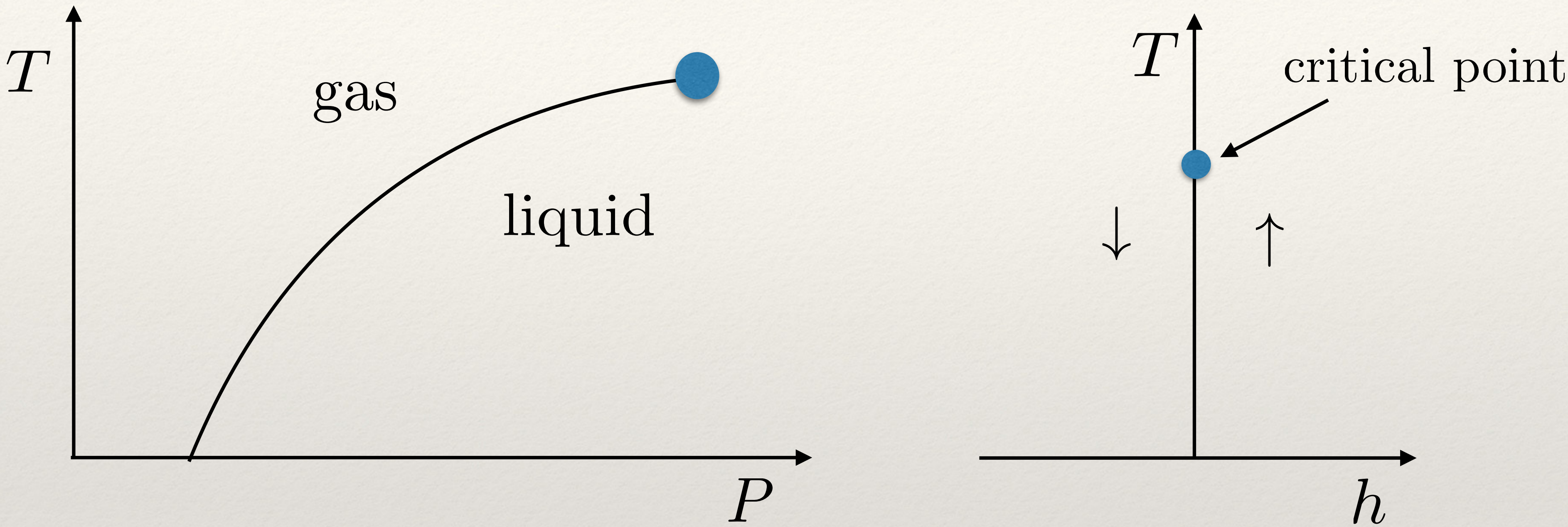
$$\langle M(r)M(0)\rangle \sim r^{-(d-2+\eta)} \qquad \Delta_M = (d-2+\eta)/2$$

Normally systems have one relevant symmetric field
- multi-critical points have more than one

Gas-liquid transition

Maps to Ising model even though no apparent Ising (Z_2) symmetry

Order parameter is density; scalar corresponds to <m> + constant



Tuning the relevant field corresponds to moving tangentially to the coexistence curve from the critical point (not so easy)

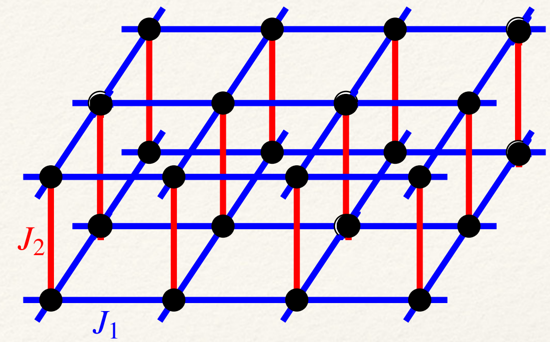
Tuning the symmetry-breaking field corresponds to moving perpendicularly to the coexistence curve

Moving along some generic path gives a mix of the two scaling dimensions in correlation functions; one eventually dominates

Example: O(3) transition in 2+1 dimensions (2D quantum)

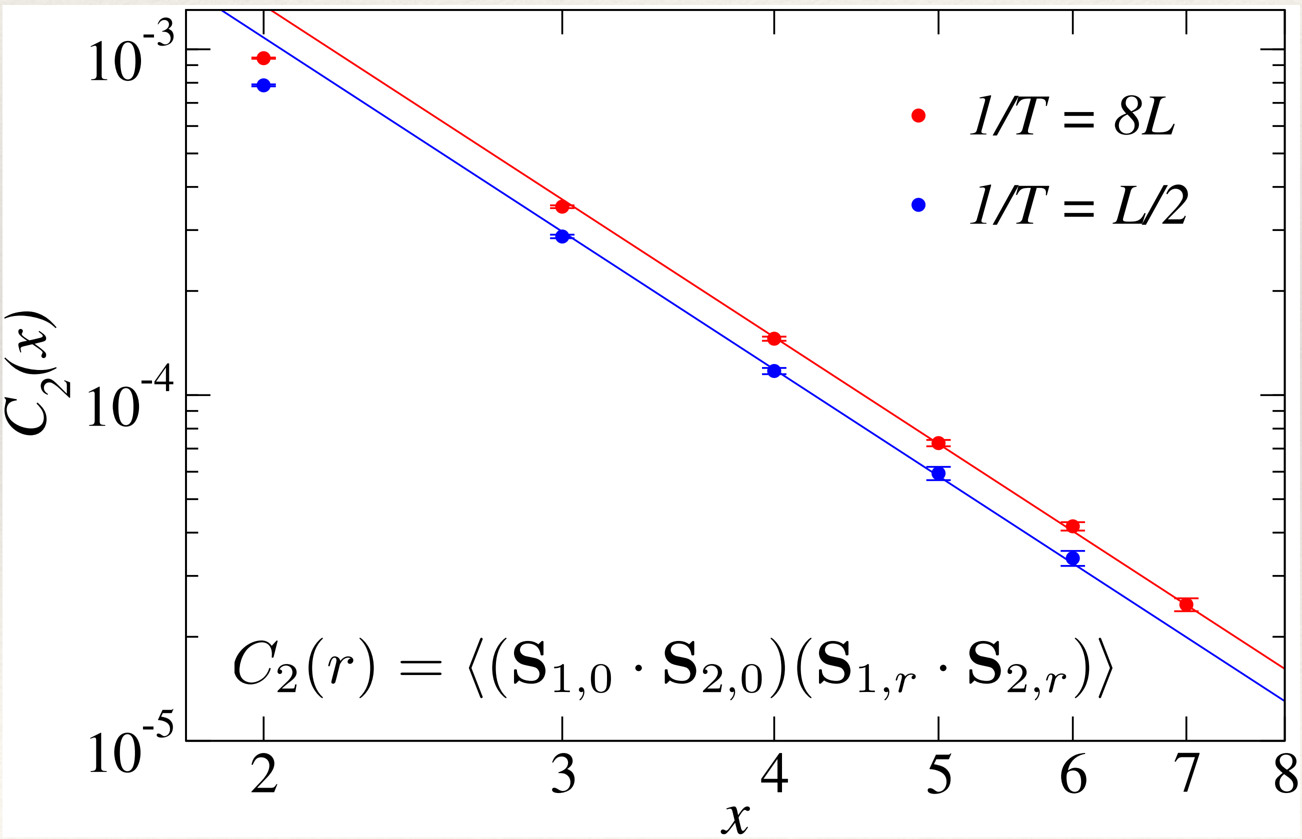
Bilayer Heisenberg model

$$H = J_1 \sum_{a=1,2} \sum_{\langle ij \rangle} \boldsymbol{S}_{a,i} \cdot \boldsymbol{S}_{a,j} + J_2 \sum_{i=1}^{N} \boldsymbol{S}_{1,i} \cdot \boldsymbol{S}_{2,i}$$



Critical at $J_2/J_1 \approx 2.5202$

The J_1 and J_2 terms are both relevant (no entropy at T=0)
- changing one of them takes us away from the critical point



$$\rightarrow \; y = d - \Delta$$

$2\Delta \approx 3.188 \rightarrow y \approx 1.406$

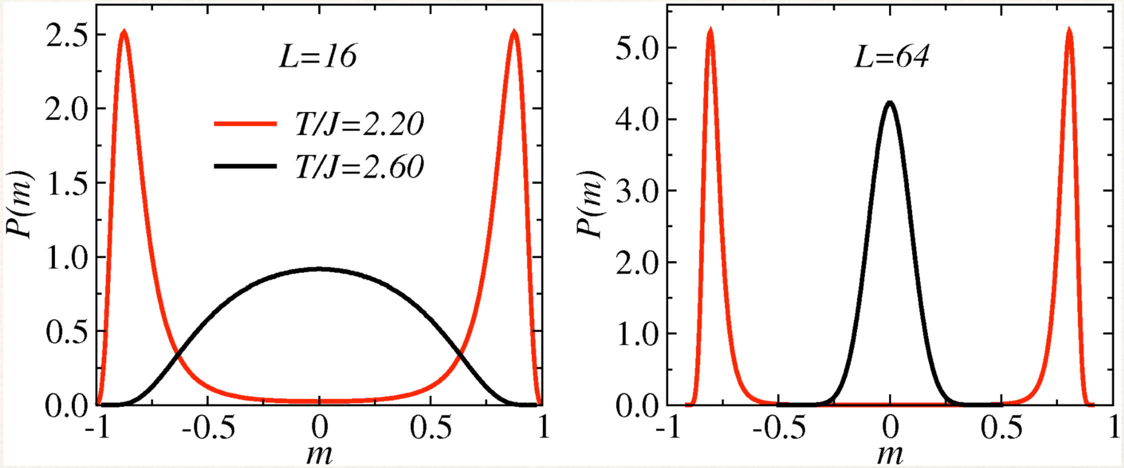
- consistent with known $1/\nu$

This is the only relevant symmetric operator at this transition

Corresponds to the classical ‘thermal field’

Binder ratios and cumulants

order parameter distribution



Consider the dimensionless ratio

$$R_2 = \frac{\langle m^4 \rangle}{\langle m^2 \rangle^2}$$

We know R_2 exactly for $N \to \infty$

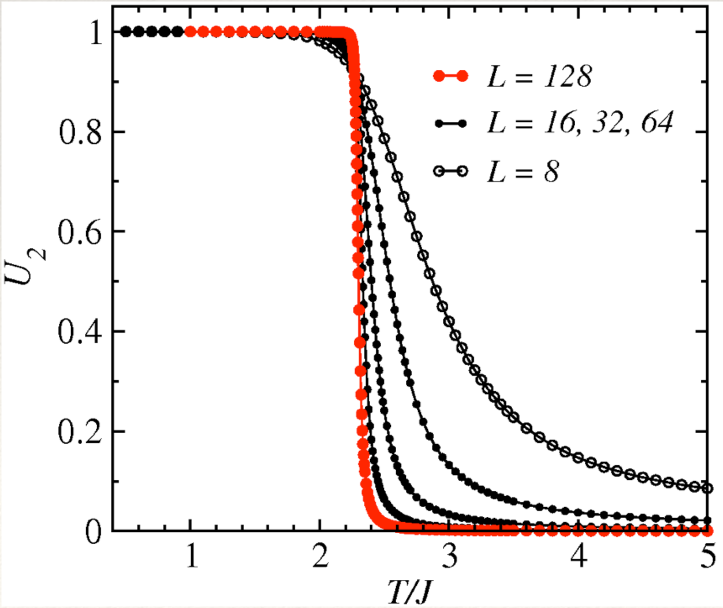
- **for $T<T_c$:** $P(m) \to \delta(m-m^*)+\delta(m+m^*)$
  m^*=|peak m-value|. **$R_2 \to 1$**

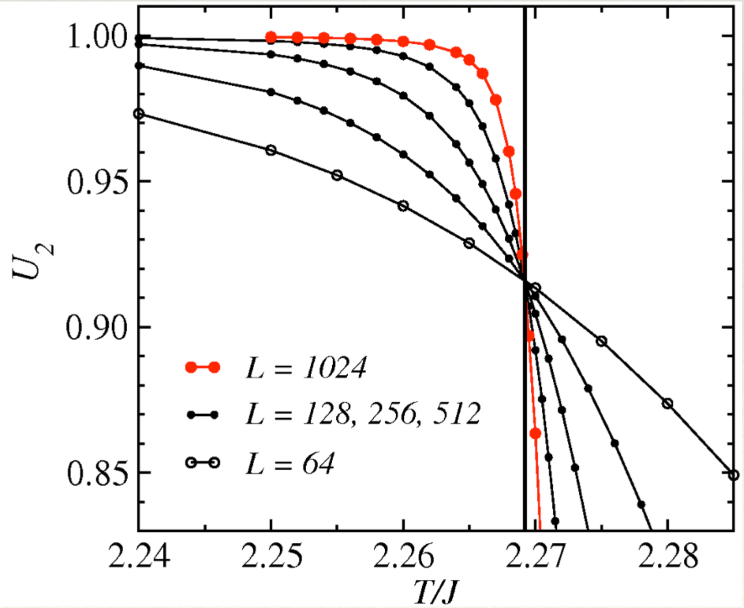
- **for $T>T_c$:** $P(m) \to \exp[-m^2/a(N)]$
  $a(N) \sim N^{-1}$ **$R_2 \to 3$** (Gaussian integrals)

The **Binder cumulant** is defined as (n-component order parameter; n=1 for Ising)

$$U_2 = \frac{n+2}{2}\left(1 - \frac{n}{n+2} R_2\right) \to \begin{cases} 1, & T < T_c \\ 0, & T > T_c \end{cases}$$

2D Ising model; MC results



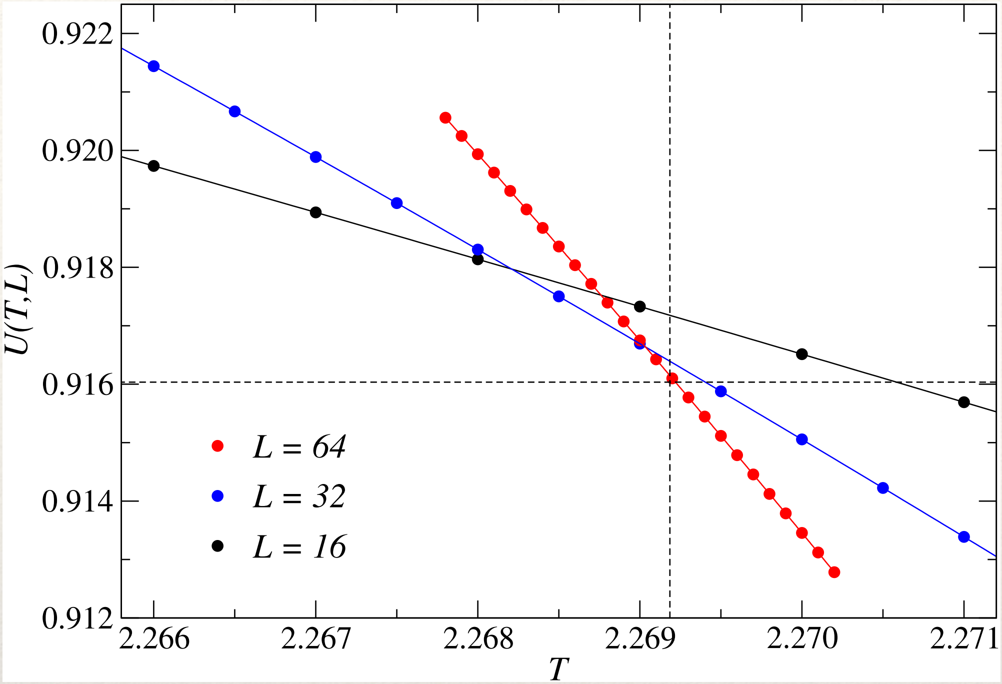


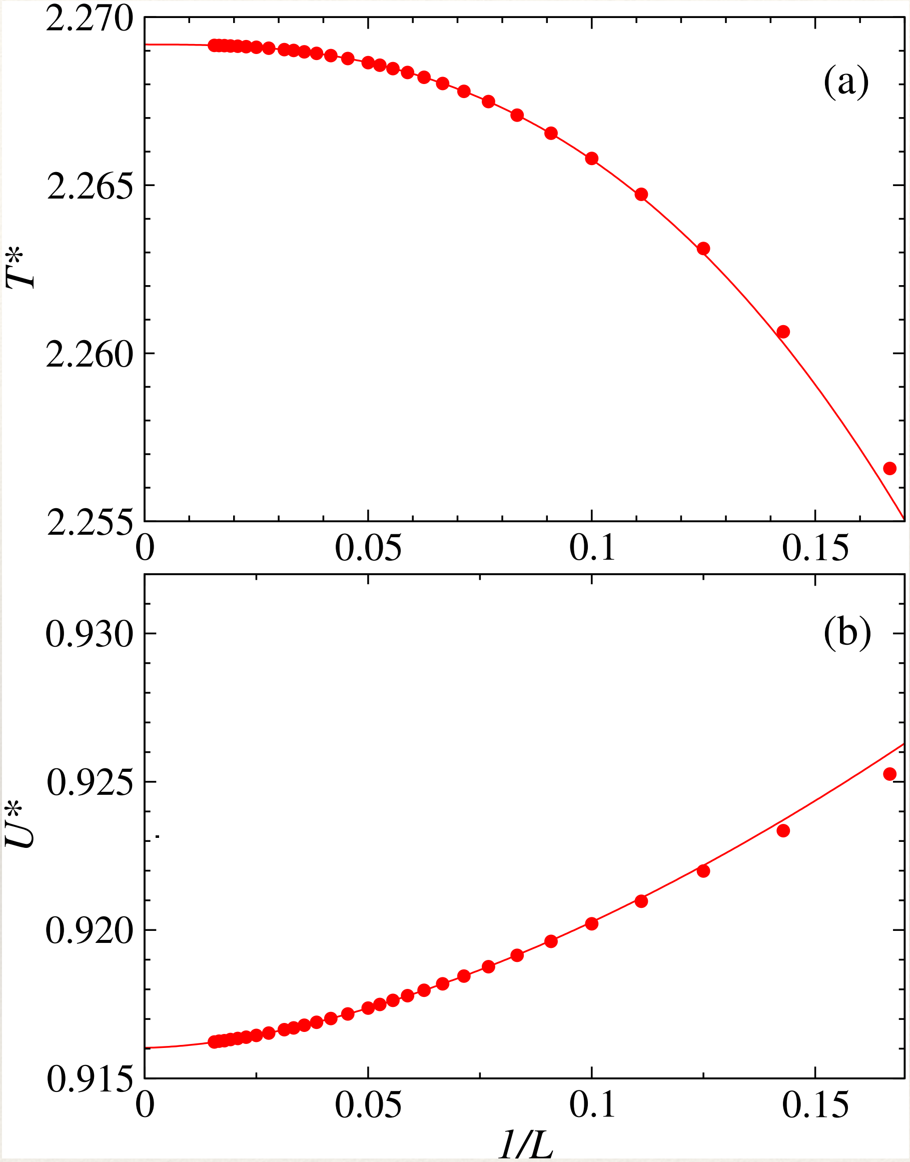
Curves for different L asymptotically cross each other at T_c

Extrapolate crossing for sizes L and 2L to infinite size

- converges faster than single-size T_c defs.

Systematic crossing-point analysis (2D Ising)





Drift in (L,2L) crossing points

$$U = U(\delta L^{1/\nu}, L^{-\omega_1}, L^{-\omega_2}, \ldots)$$

⇒ scaling corrections in crossings

$\sim L^{-(1/\nu+\omega)}$ for $T^* \rightarrow T_c$

$\sim L^{-\omega}$ for $U^* \rightarrow U(T_c)$

Use correction with free exponent

Fit with L_{min}=12: T_c=2.2691855(5). Correct: T_c=2.2691853...

Correlation-length exponent

Consider some generic critical observable A

$$A(L,t) = L^{-\kappa/\nu} f(\delta L^{1/\nu}) \;\rightarrow A(L,t)L^{\kappa/\nu} = f(\delta L^{1/\nu})$$

Let us take the derivative wrt δ

$$\frac{df(\delta L^{1/\nu})}{d\delta} = L^{1/\nu} f'(\delta L^{1/\nu}) \quad \rightarrow \frac{d(AL^{\kappa/\nu})}{d\delta} \propto L^{1/\nu} \quad (\delta = 0)$$

The Binder cumulant is dimensionless

$$U = U(\delta L^{1/\nu}, L^{-\omega_1}, L^{-\omega_2}, \ldots)$$

$$\frac{1}{\ln(2)} \ln\left(\frac{U'(2L)}{U'(L)}\right) \rightarrow \frac{1}{\nu}$$

Test for 2D Ising (ν=1)