Dangerously Irrelevant Fields, Two Length Scales , And Monte Carlo Renormalization Flows of the Clock Models

Hui Shao

Jan 2021, BNU

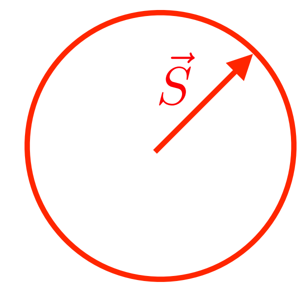
Outline

- **3D classical XY model:**
  - general understanding.
- **3D classical clock model:**
  - dangerously irrelevant field and two length scales;
  - Monte Carlo renormalization flows.
- **2D quantum clock model:**
  - Same critical properties;
  - different U(1) to Z(q) scaling behavior;
  - mapping to 3D classical anisotropic clock model;
- **Conclusions.**

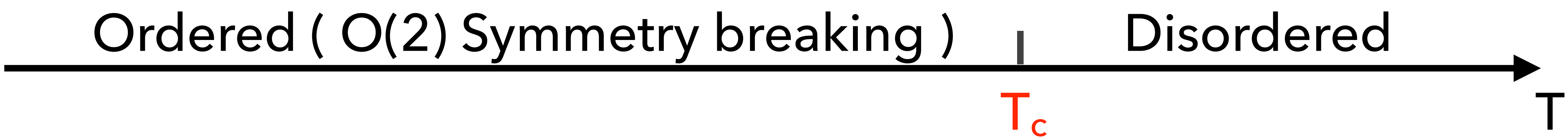
3D classical XY model

- Hamiltonian:

$$H = -\sum_{\langle i,j \rangle} \vec{S}_i \cdot \vec{S}_j = -\sum_{\langle i,j \rangle} \cos\left(\theta_i - \theta_j\right)$$



- Phase digram:

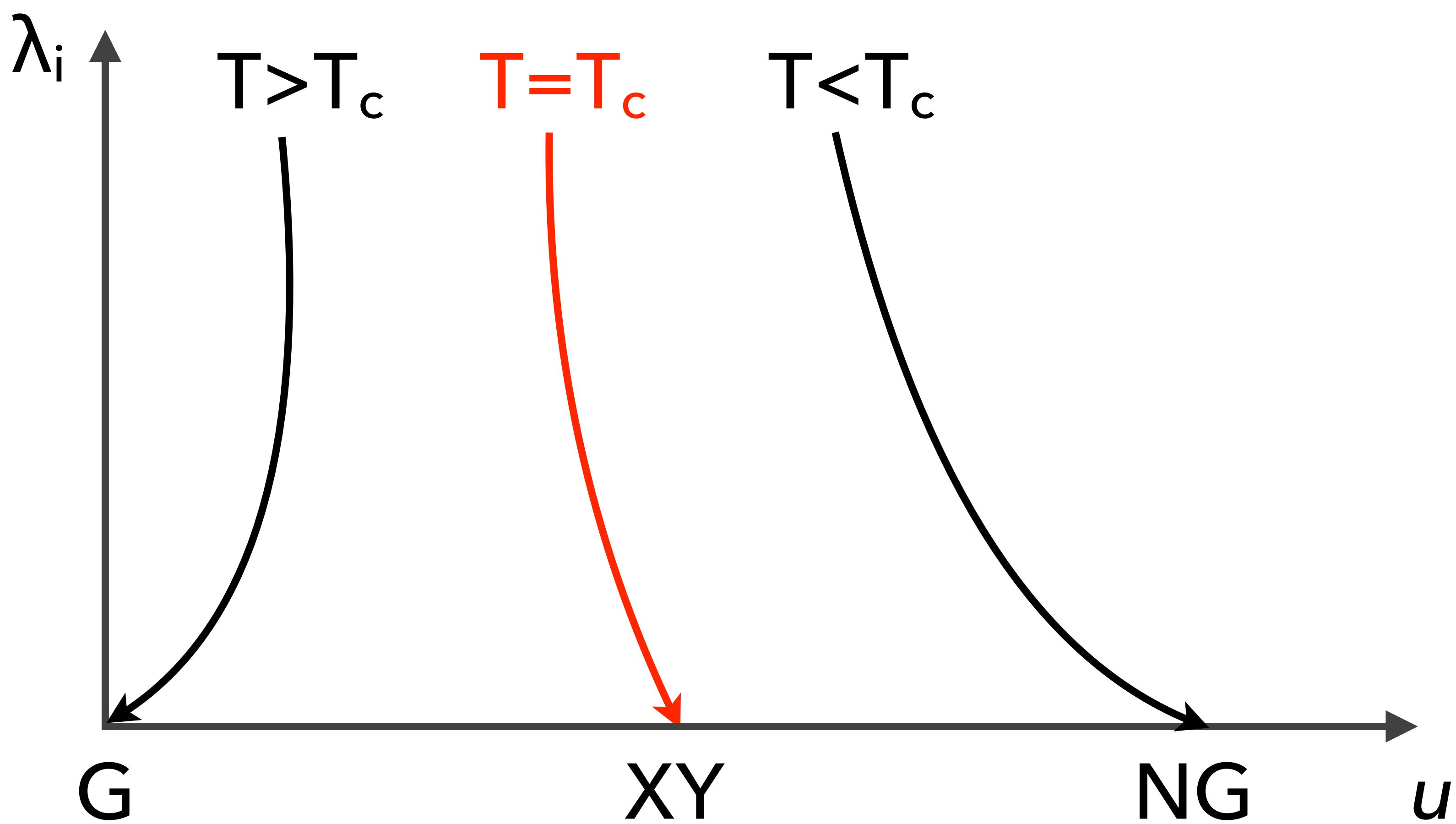


- Low energy effective model:

  - In the disordered phase, $H(\Phi)/T = \int d^3x \left[|\partial_\mu \Phi|^2 + u|\Phi|^2 \right]$
  - Around T_c, $H(\Phi)/T = \int d^3x \left[|\partial_\mu \Phi|^2 + u|\Phi|^2 + g|\Phi|^4 \right]$
  - In the ordered phase, $H(\theta)/T = \int d^3x \left(\partial_\mu \theta\right)^2 /2$

3D classical XY model

- RG picture:



Three fixed points:

- $T>T_c$, Gaussian fixed point (G);
- $T=T_c$, XY universality class fixed point (XY);
- $T<T_c$, Nambu-Goldstone fixed point (NG).

3D classical XY model

- Different calculations of the critical exponents:

Method	Reference	ν	η
perturbation series at fixed dimension including seven–loop contributions	J. Phys. A 31, 8103 (1998)	0.6703(15)	0.0354(25)
ε-expansion at order ε^5	J. Phys. A 31, 8103 (1998)	0.6680(35)	0.0380(50)
first-order non-perturbative RG	Phys. Rept. 363, 223 (2002)	0.704	0.049
Experiments on helium	Low Temp. Phys.93, 131 (1992)	0.6705(6)	
high temperature expansions+MC	PRB 74, 144506 (2006)	0.6717(1)	0.0381(2)
Monte Carlo	PRB 100, 224517 (2019)	0.67169(7)	0.03810(8)
conformal bootstrap	JHEP 06, 142 (2020)	0.67175(1)	0.038176(44)

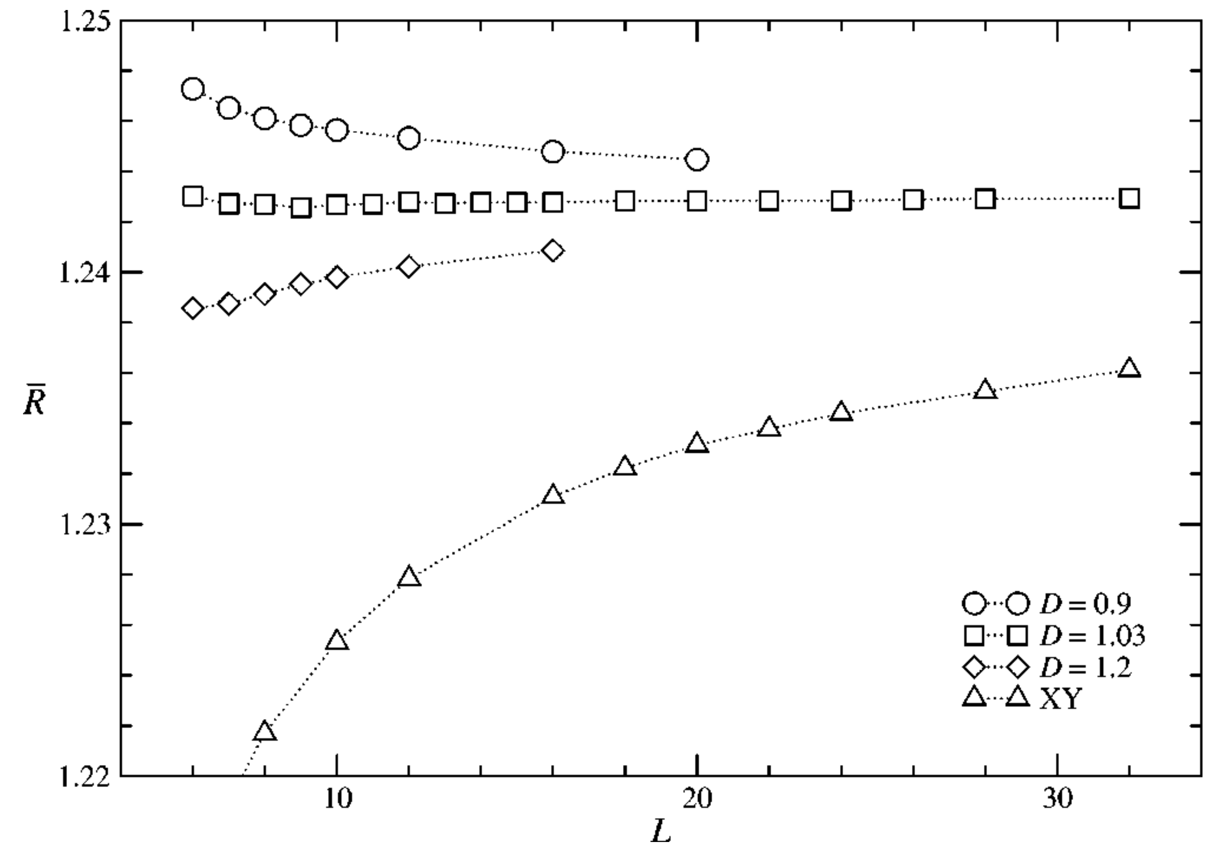
3D classical XY model

- Monte Carlo Simulations:

  - Corrections exist in simulations of the standard XY model:

$$R = R\left(\delta L^{1/\nu}, L^{-\omega}, \dots\right)$$

  leading order $\omega = 0.785(20)$.



- Adding a term corresponding to the leading correction field:

$$H = -\beta \sum_{\langle i,j \rangle} \vec{S}_i \cdot \vec{S}_j - D \sum_i \vec{S}_i^2$$

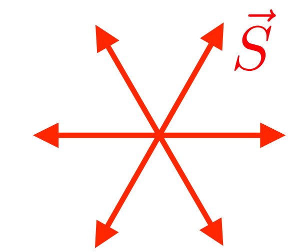
ordered phase and the phase transition not changed; corrections being eliminated.

Campostrini et. al., PRB (2001).

3D classical clock model

- Hamiltonian:

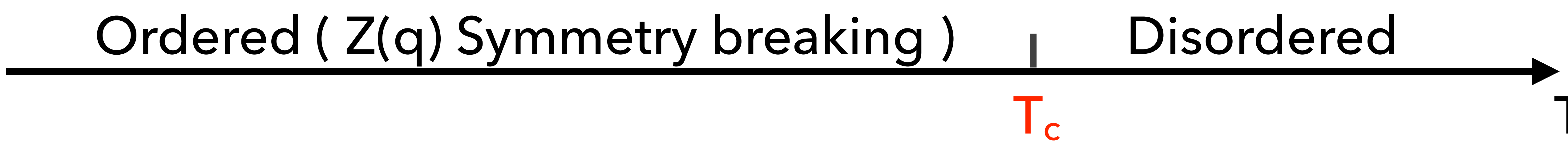
$$H = -\sum_{<i,j>} \cos(\theta_i - \theta_j) - h\sum_i \cos(q\theta_i)$$



In the h=0 or $q = \infty$ limit, back to XY model;

In the $h = \infty$ limit, XY interaction with $\vec{S}$ pointing to q directions.

- Phase digram For q>=4:



- the phase transition belongs to 3D XY universality class.

- Z(q) field is irrelevant at the critical point but relevant in the ordered phase – Dangerously Irrelevant Field.

Nelson, PRB (1976).

3D classical clock model

- Perturbation of the effective model

  - At critical point:

$$S = \int d^3x \left[|\partial_\mu \Phi|^2 + u|\Phi|^2 + g|\Phi|^4 - \lambda_q \left(\Phi^q + \bar{\Phi}^q \right) \right]$$

  Scaling dimension of λ_q based on ε-expansion,

$$y_q = 4 - q + \epsilon \left(\frac{q}{2} - 1 - \frac{q(q-1)}{10} \right) + O\left(\epsilon^2\right)$$

  $y_q<0$ when $q>=4$, λ_q is irrelevant.

  - In the ordered phase:

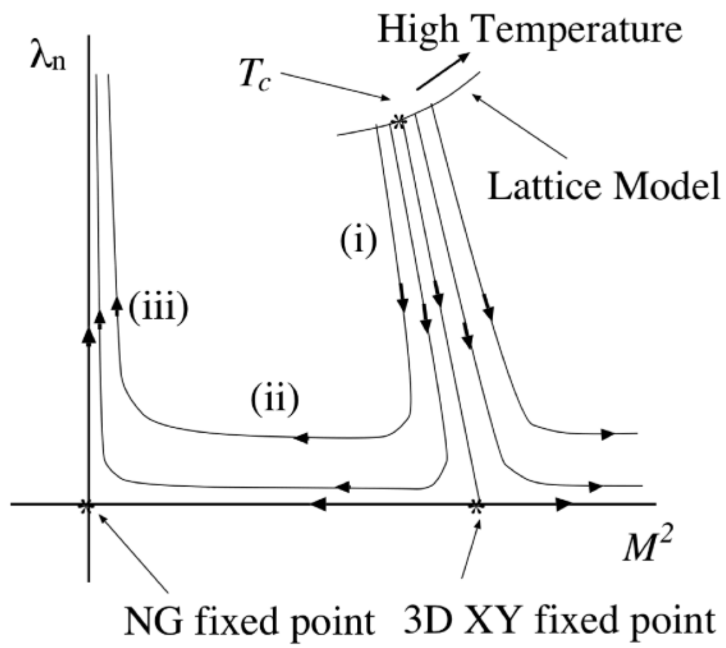
$$\mathrm{D} = \int d^3x \frac{1}{2} \left(\partial_\mu \theta\right)^2 - \lambda_q K^3 \int d^3x \cos\left[q \left(\phi_0 + \frac{\theta}{\sqrt{K}} \right) \right]$$

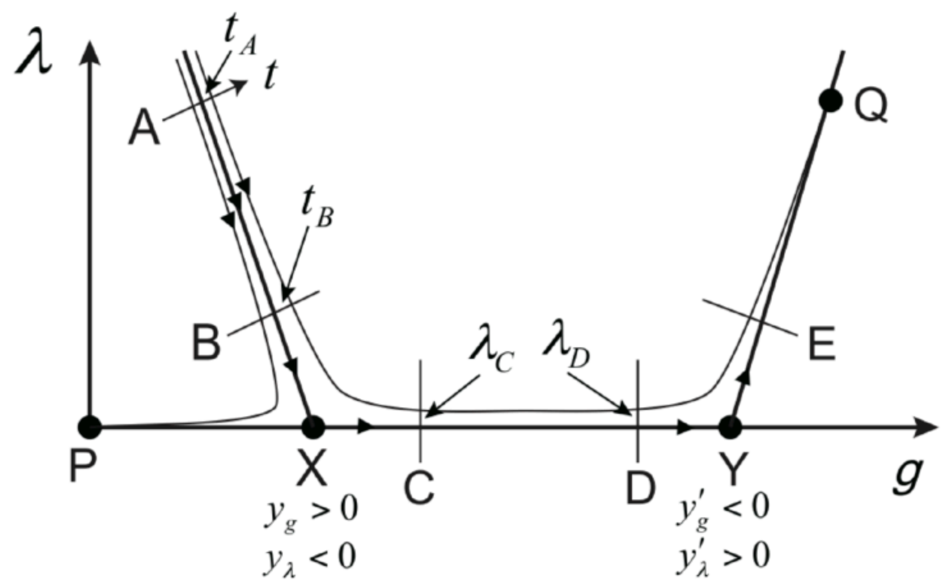
  Taylor expansion indicates the relevance of the Z(q) field for any value of q.

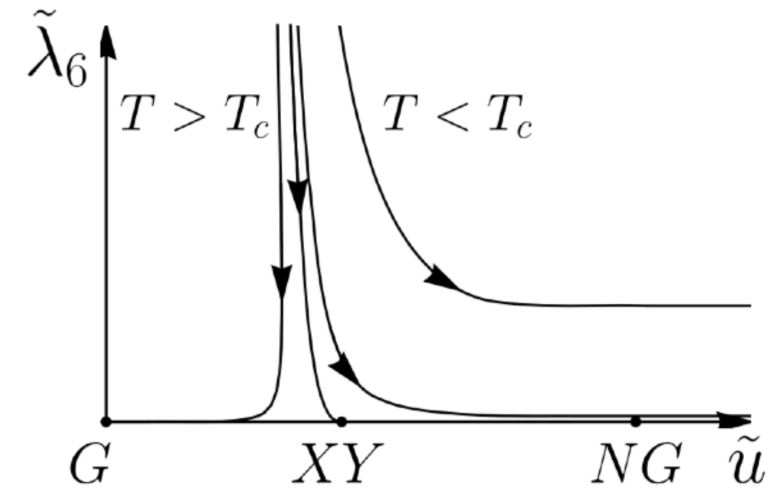
Oshikawa, PRB (2000).

3D classical clock model

- RG flow







- Crossover from the NG fixed point to the Z(q) symmetry breaking fixed point;

- A larger length scale: $\xi' \sim t^{-\nu'}$, when $\xi \ll L \ll \xi'$, the system looks U(1) ordered;

- Different proposals of the scaling relation with ν, ν', y_q .

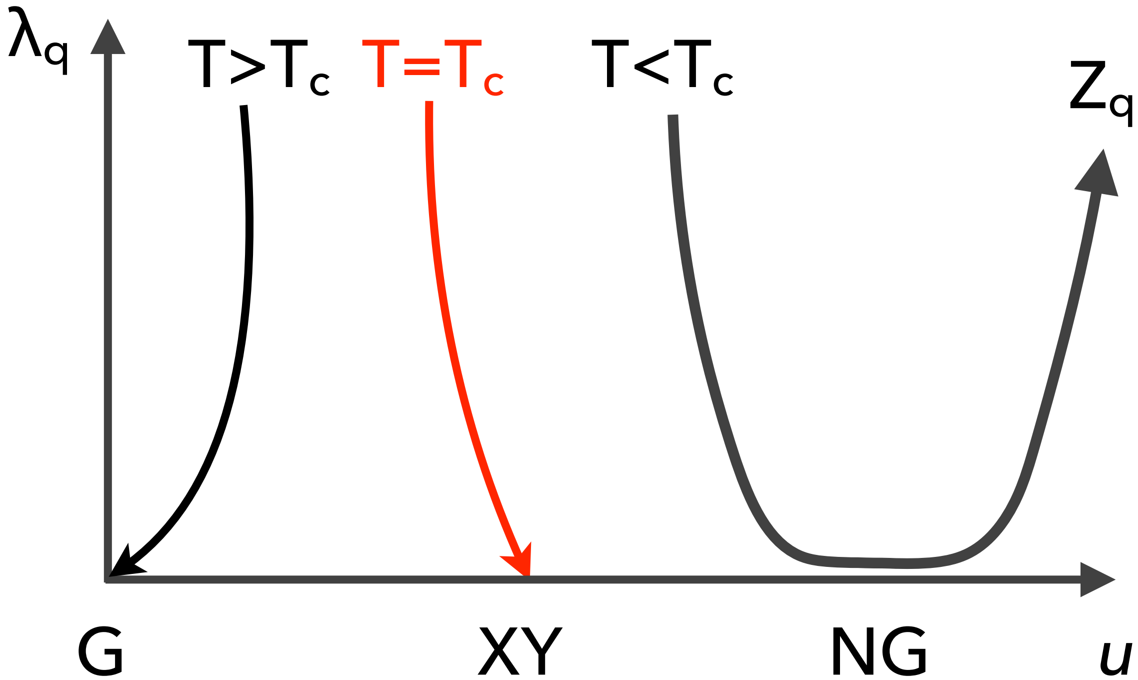
Oshikawa, PRB (2000). Okubo et. al., PRB (2015). Lonard et. At., PRL (2015).

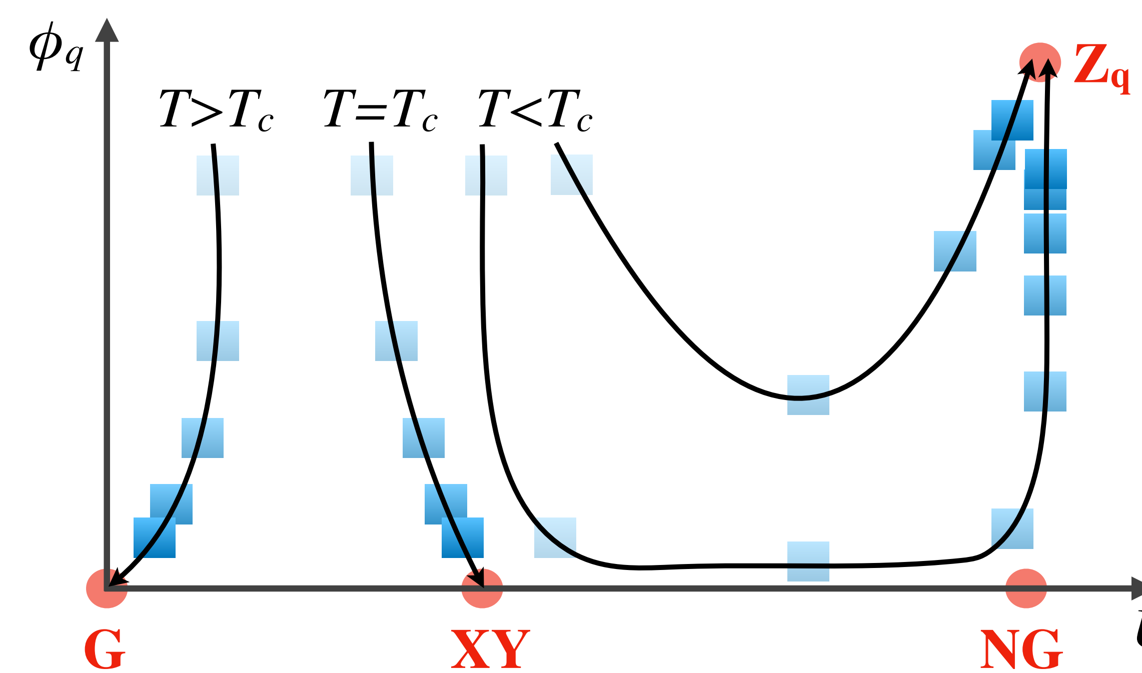
3D classical clock model

- Monte Carlo renormalization flows:

  - Works in the space of physical observables corresponding to the given fields . By increasing the system size, observables approach their thermal dynamic values according to the scaling exponents Δ .

  - Finite size scaling can be considered as an example.





Shao , Guo and Sandvik, PRL (2020). Patil , Shao, Sandvik , arXiv:2009.03249 (2020).

3D classical clock model

- Corresponding physical observables:

Temperature

Binder cumulant

$$U = 2 - \langle m^4 \rangle / \langle m^2 \rangle^2$$

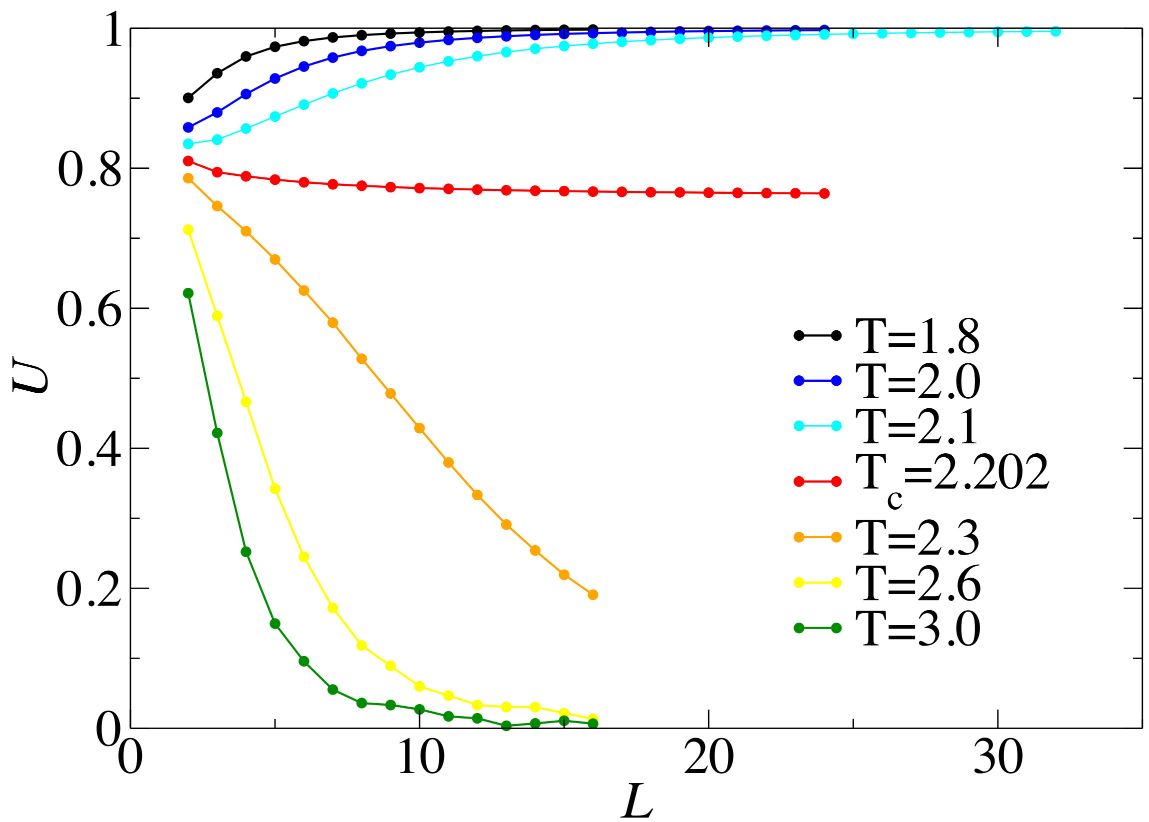
Z(q) field

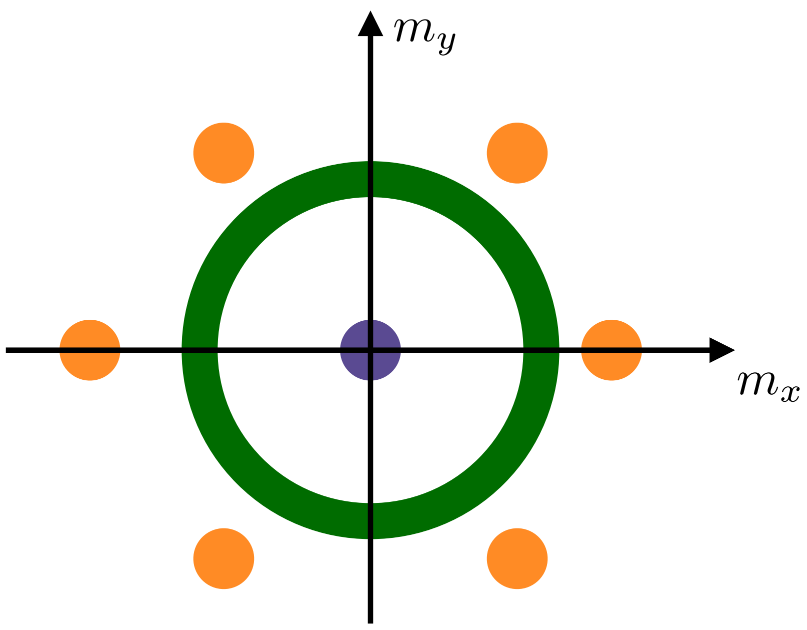
Angular order parameter

$$\phi_q = \langle \cos(q\theta) \rangle$$

$$\theta = \arccos(m_x / \sqrt{m_x^2 + m_y^2})$$

Space averaged!





3D classical clock model

- Corresponding physical observables:

Temperature

Binder cumulant

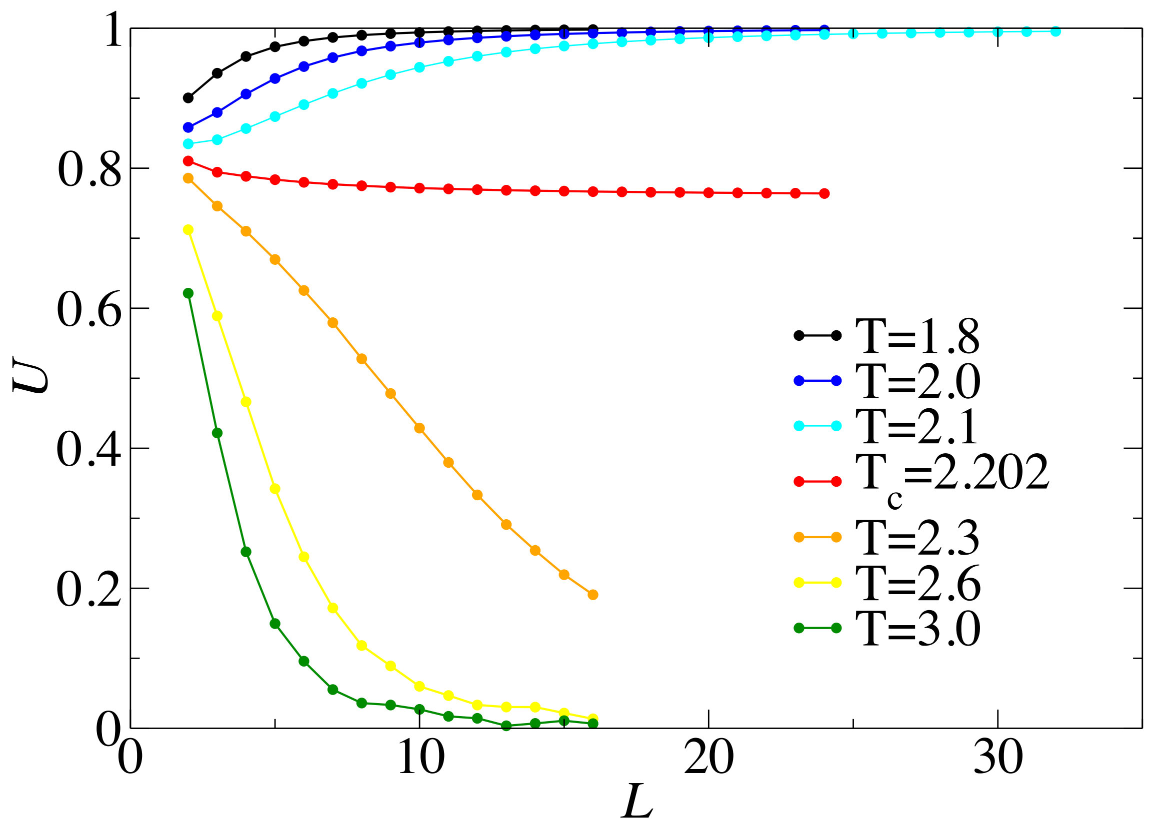
$$U = 2 - \langle m^4 \rangle / \langle m^2 \rangle^2$$

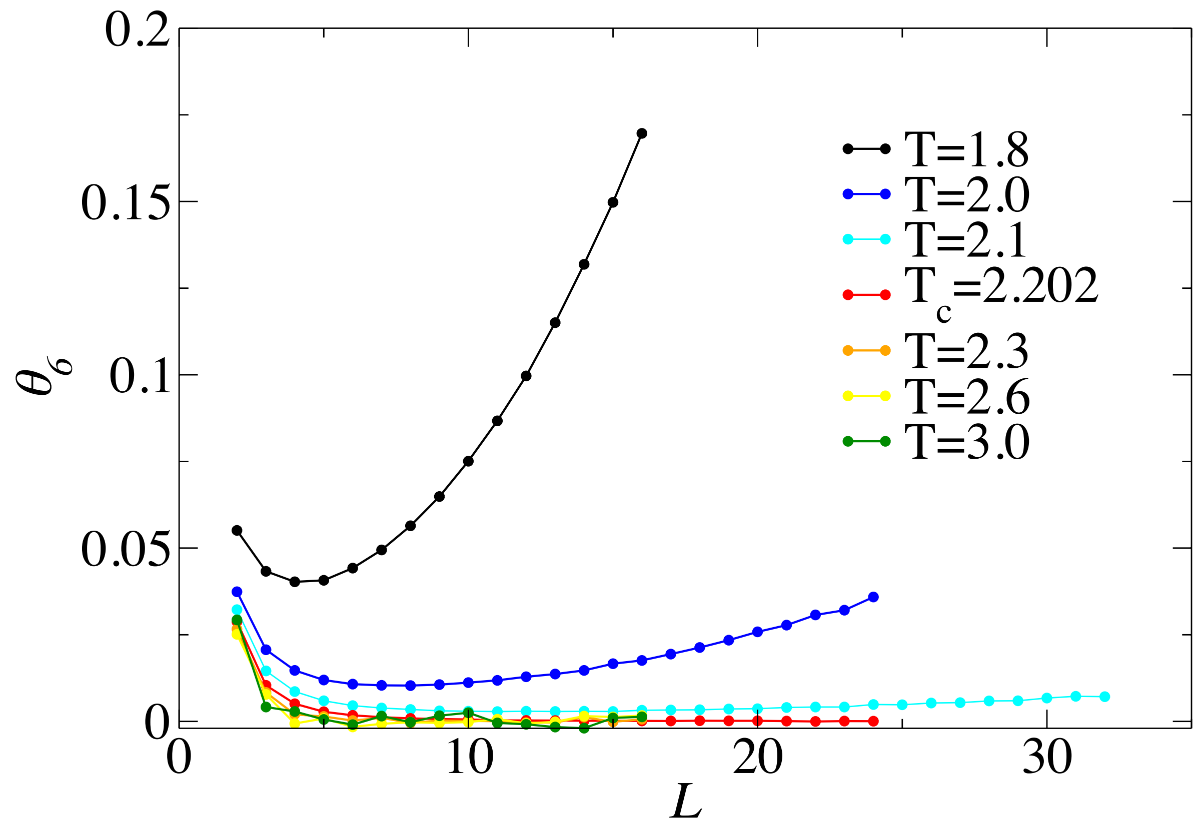
Z(q) field

Angular order parameter

$$\phi_q = \langle \cos(q\theta) \rangle$$

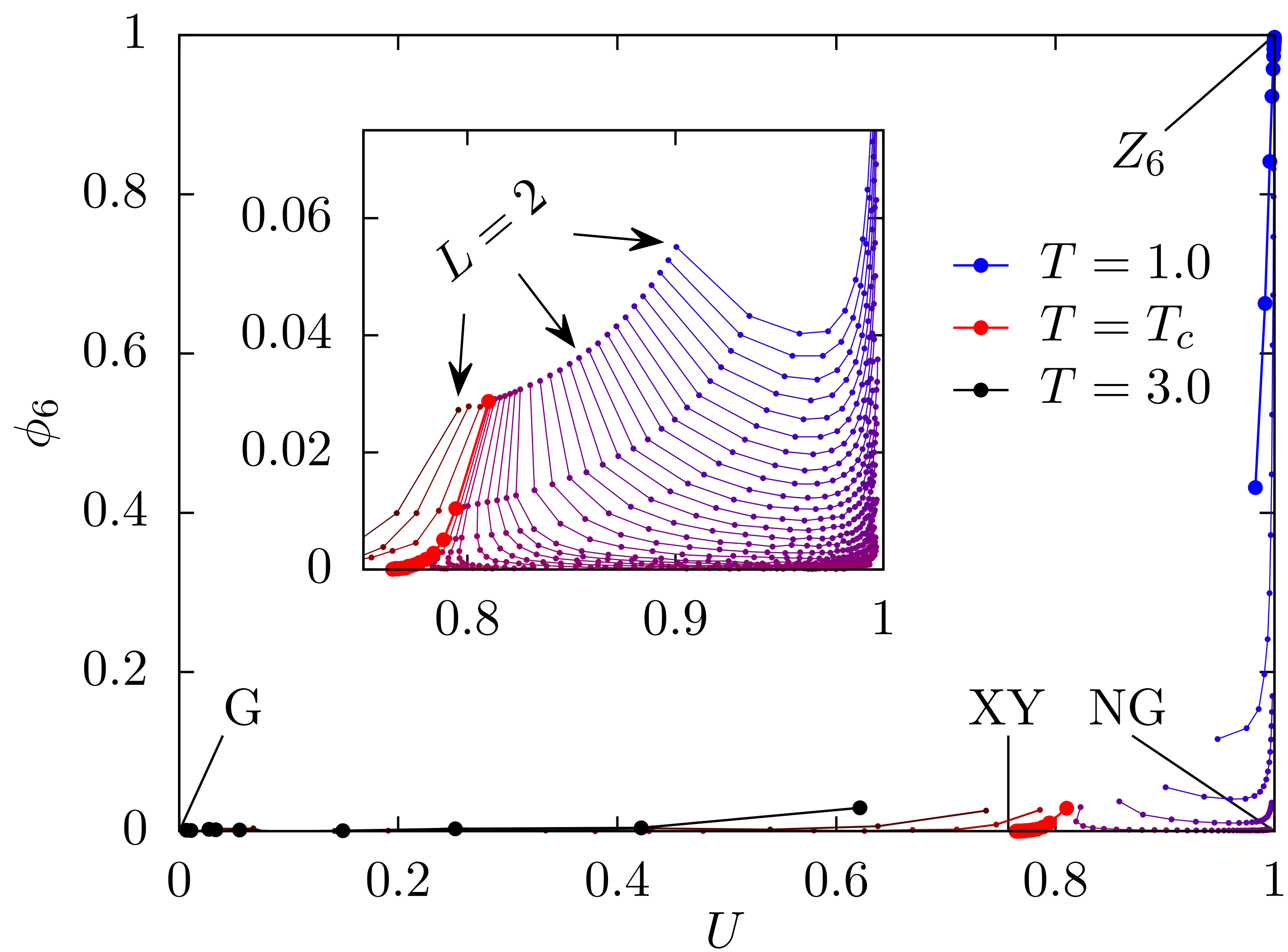
$$\theta = \arccos(m_x / \sqrt{m_x^2 + m_y^2})$$





3D classical clock model

- Flow Picture - clear illustration of all the renormalization stages.



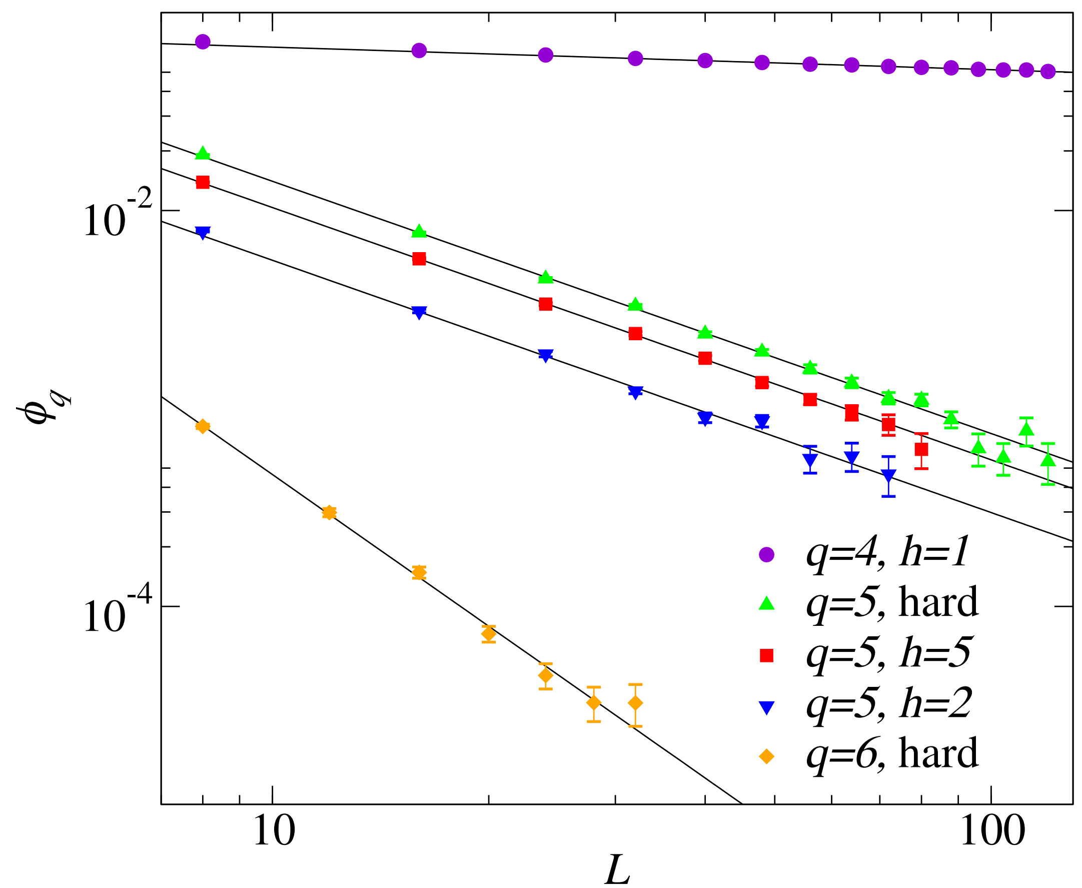
- Valid two length scale hypothesis:

$$\phi_q \sim L^{-|y_q|}\Phi(tL^{1/\nu}, tL^{1/\nu_q'})$$

Shao , Guo and Sandvik, PRL (2020).

3D classical clock model

- Scaling dimension of the Z(q) field at T$_c$, $\phi_q \sim L^{-\Delta} \sim L^{y_q}$



q / $-y_q$	4	5	6
ε expansion	0.2	1.5	3.0
non-perturbative RG	0.114	1.16	2.29
standard calculation +extrapolation	0.108(6)	1.25	2.5
MC of CFT	0.128(6)	1.265(6)	2.509(7)
This work	0.114(2)	1.27(1)	2.55(6)
Conformal bootstrap	0.14(2)		

Shao , Guo and Sandvik, PRL (2020).

3D classical clock model

- Scaling dimension of the Z(q) field at T_c

  - standard way: correlation function of the local operator at the transition point of XY model.

  Local operator:

$$m(q, r_i) = \cos(q\theta_i)$$

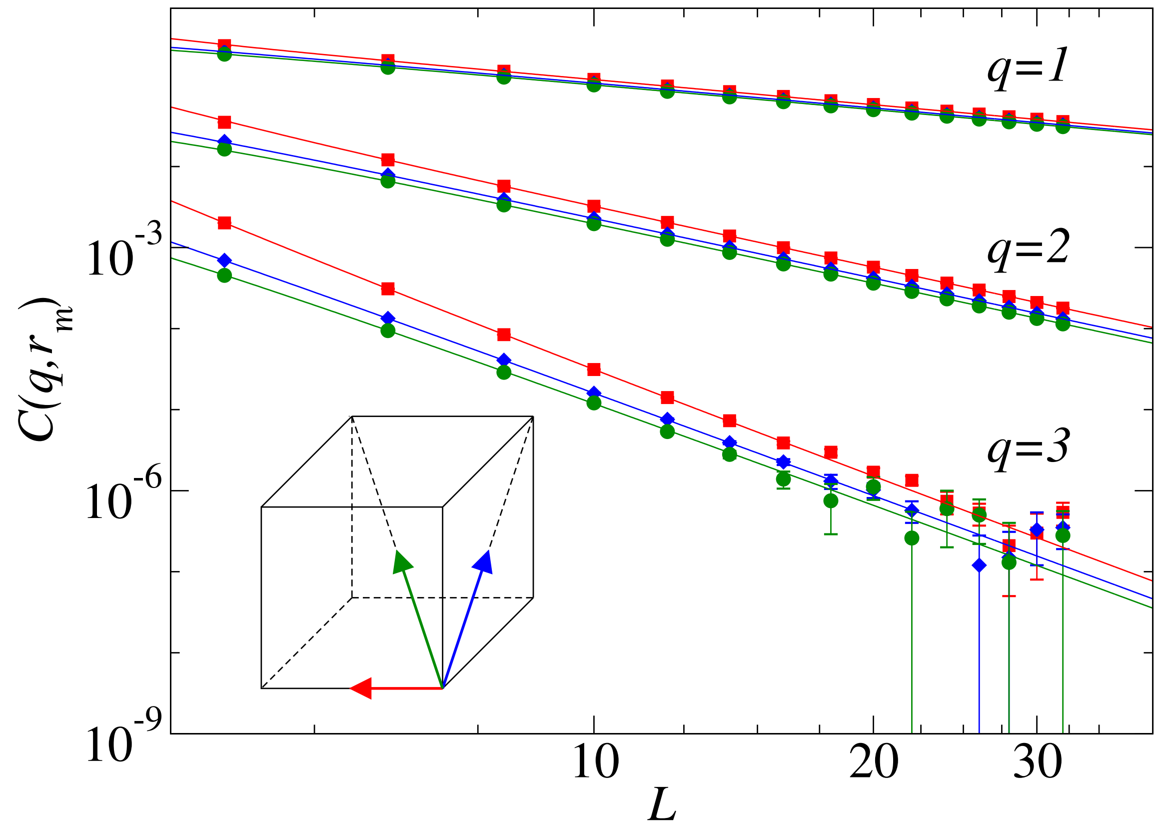
  Correlation function:

$$C(q,r) = \langle m(q,r_i) m(q,r_j) \rangle$$

$$= \langle \cos(q\theta_i - q\theta_j) \rangle$$

  Scaling behavior:

$$C(q, r_{\mathrm{m}}) \sim a L^{-2\Delta_q}(1 + bL^{-\omega})$$



  Scaling dimension of the Z(q) field: $y_q = 3 - \Delta_q$

Shao , Guo and Sandvik, PRL (2020).

3D classical clock model

- Scaling dimension of the Z(q) field at T_c

  - standard way: correlation function of the local operator at the transition point of XY model.

  Local operator:

  $$m(q, r_i) = \cos(q\theta_i)$$

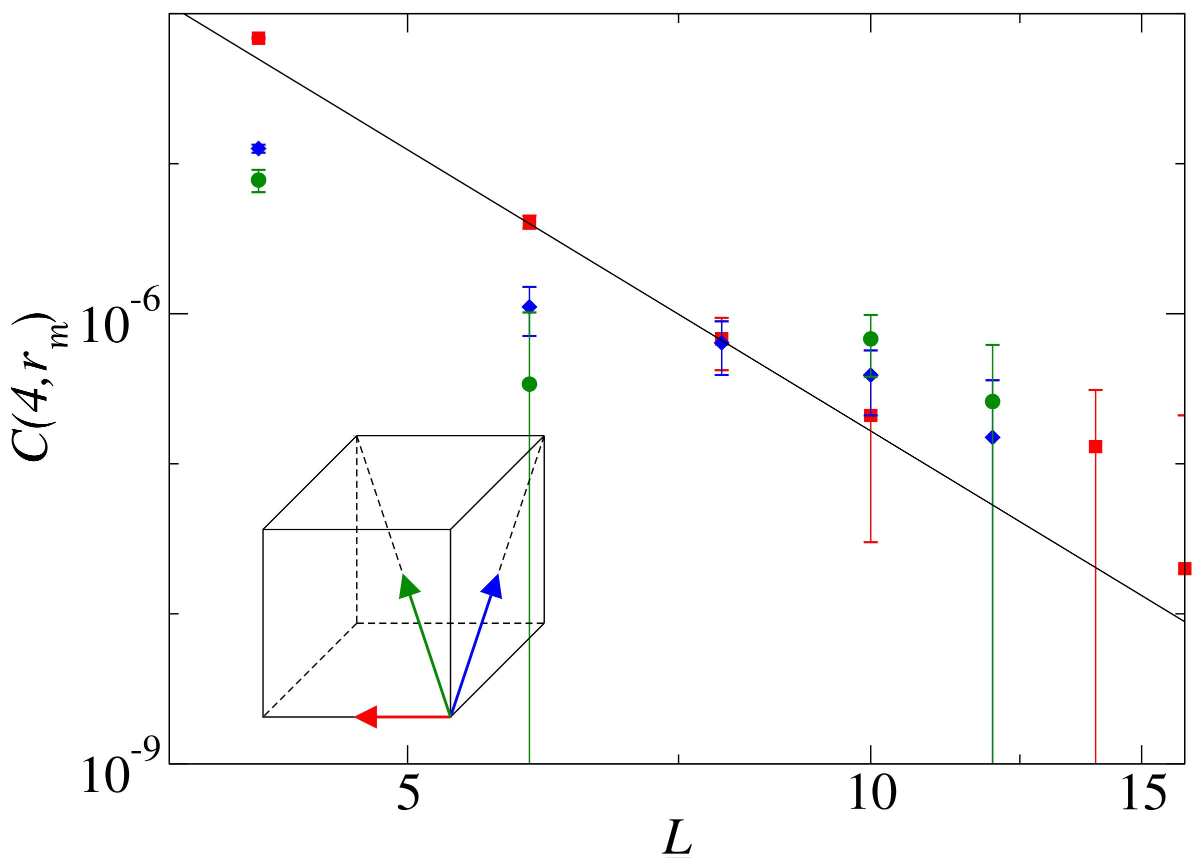
  Correlation function:

  $$C(q,r) = \langle m(q,r_i) m(q,r_j) \rangle$$

  $$= \langle \cos(q\theta_i - q\theta_j) \rangle$$

  Scaling behavior:

  $$C(q, r_{\mathrm{m}}) \sim a L^{-2\Delta_q} (1 + bL^{-\omega})$$



Scaling dimension of the Z(q) field: $y_q = 3 - \Delta_q$

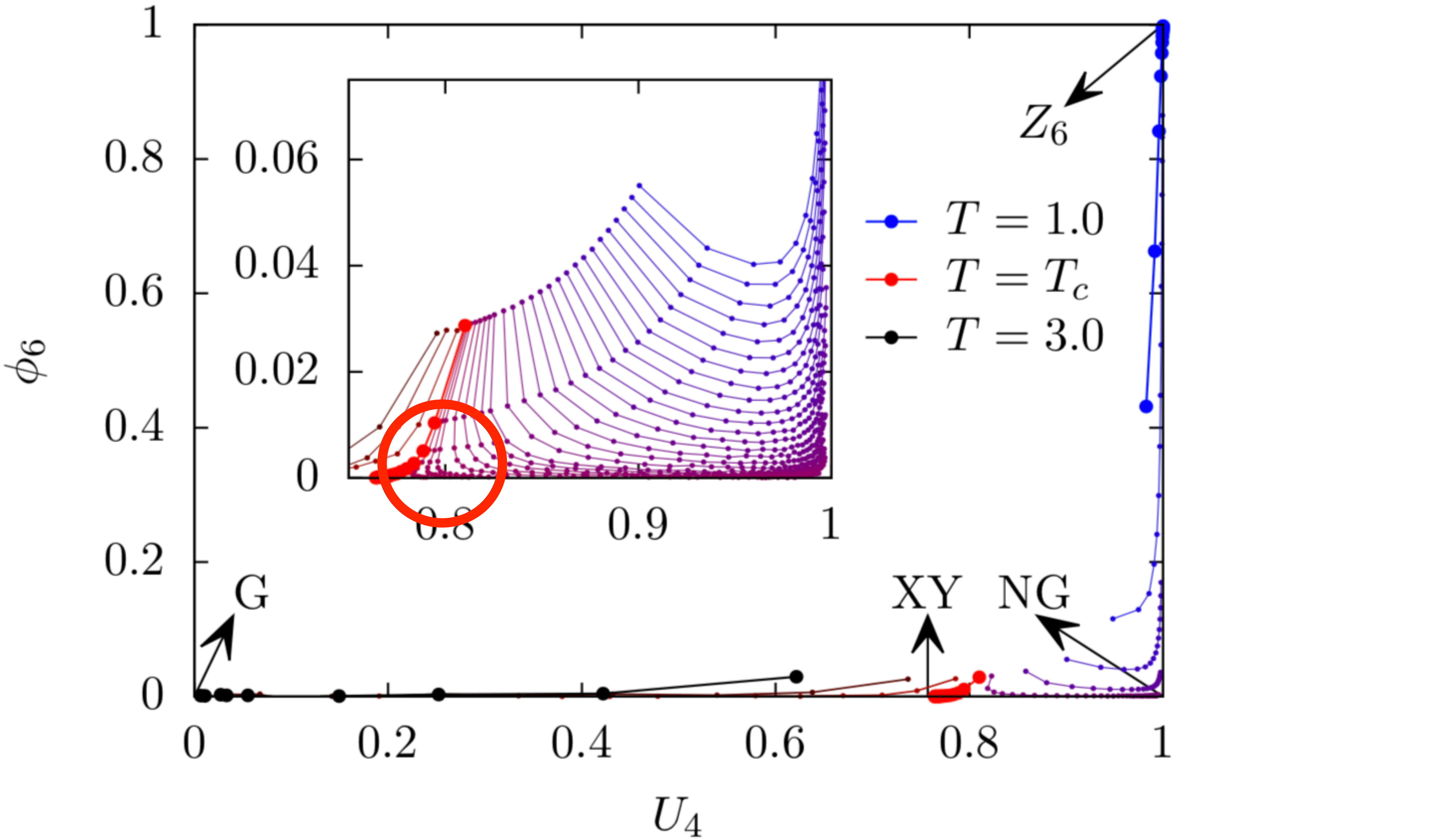
$2\Delta_q > 6$ for irrelevant field, extremely hard to extrapolate.

Shao , Guo and Sandvik, PRL (2020).

3D classical clock model

- When $tL^{1/\nu'_q} \ll tL^{1/\nu} \ll 1$, dominated by the XY fixed point

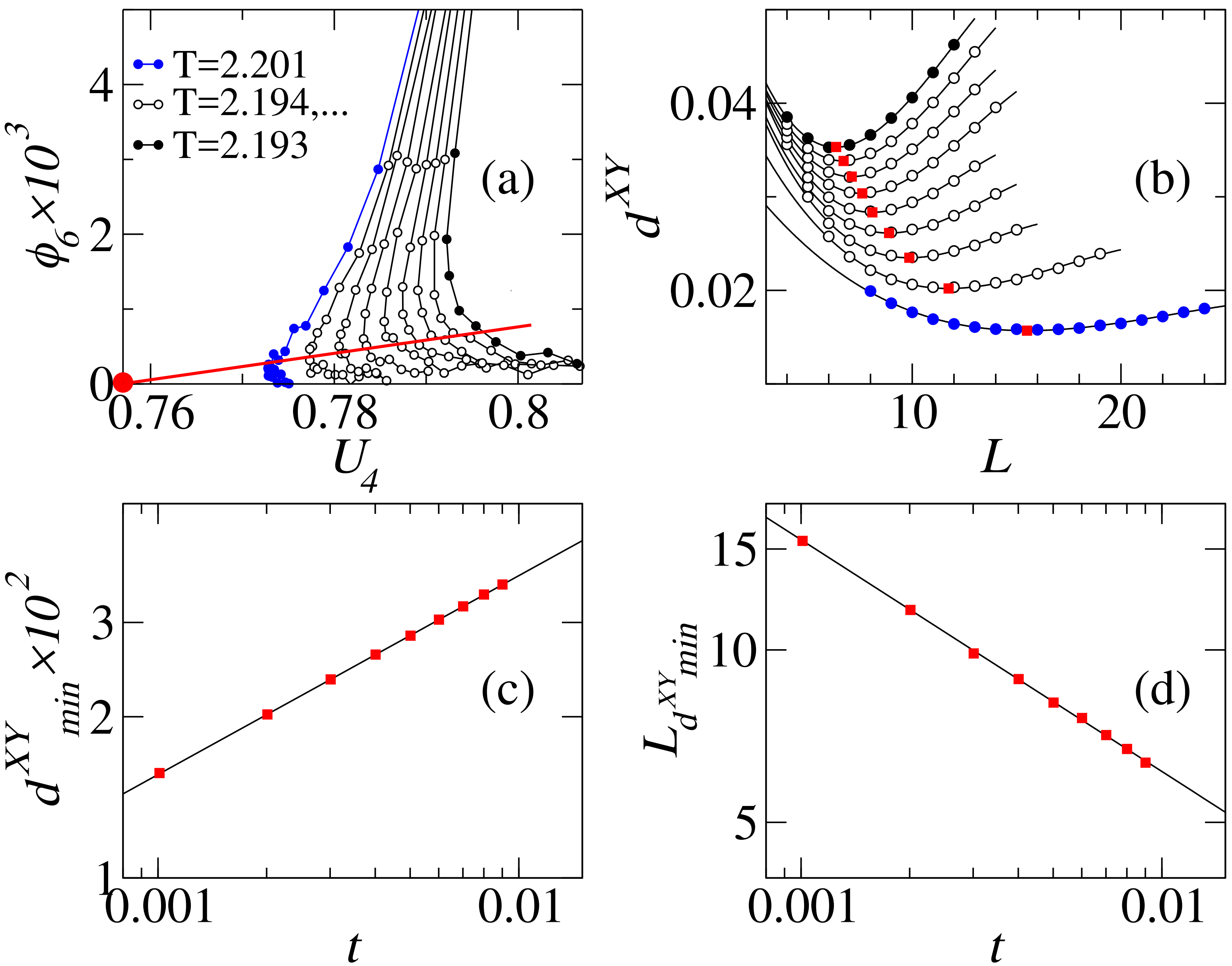
$$\phi_q \propto L^{y_q}(1 + tL^{1/\nu}) \qquad U = U(tL^{1/\nu}) = U_{XY} + tL^{1/\nu} + L^{-\omega}$$



3D classical clock model

- When $tL^{1/\nu_q'} \ll tL^{1/\nu} \ll 1$, dominated by the XY fixed point

  Distance to the XY fixed point: $d_{XY} \propto \sqrt{\left(tL^{1/\nu} + L^{-\omega}\right)^2 + L^{2y_q}\left(1 + tL^{1/\nu}\right)^2}$



$$d^{XY}_{min} \propto t^{\frac{\omega}{1/\nu+\omega}} \propto t^{0.38(2)}$$

$$L_{d^{XY}_{min}} \propto t^{\frac{-1}{1/\nu+\omega}} \propto t^{-0.408(5)}$$

$$d^{XY}_{min} \sim t^{0.372(1)}$$

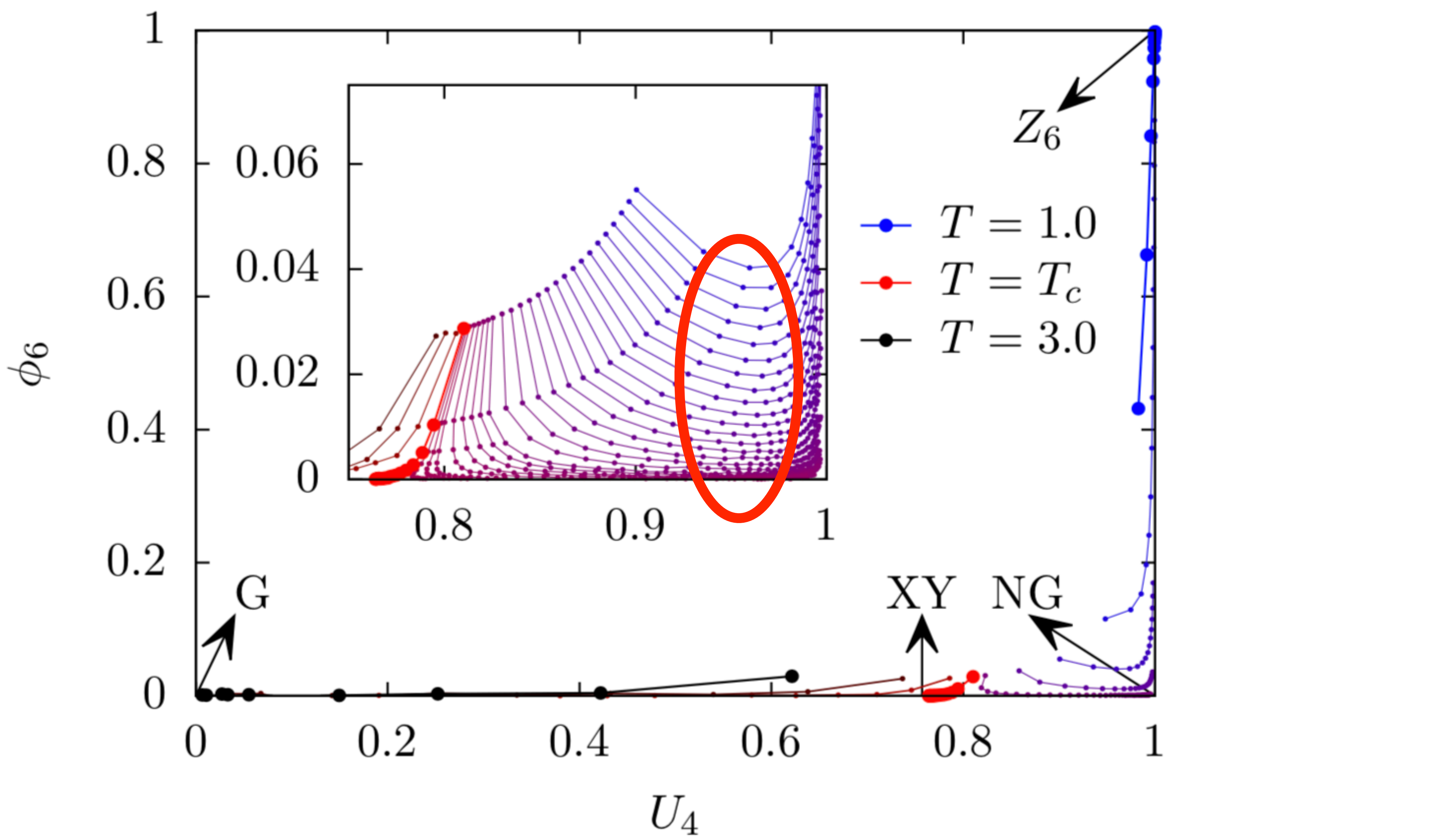
$$L_{d^{XY}_{min}} \sim t^{-0.404(4)}$$

Shao , Guo and Sandvik, PRL (2020).

3D classical clock model

- When $tL^{1/\nu}$ grows and $tL^{1/\nu'_q} \ll 1$, still dominated by the XY fixed point (in principle higher orders should be included)

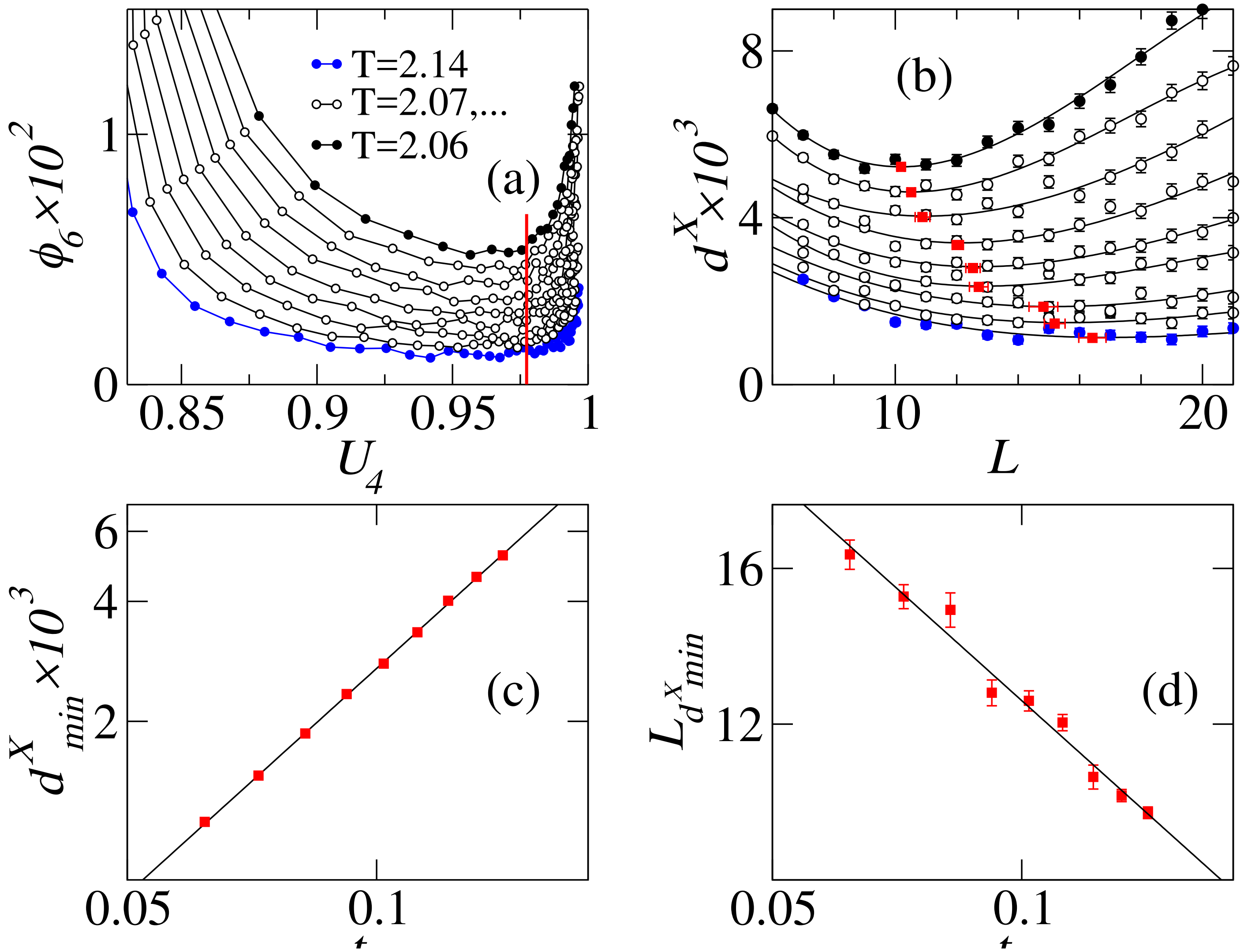
$$\phi_q \propto L^{y_q}(1 + tL^{1/\nu}) \qquad U = U(tL^{1/\nu}) = U_{XY} + tL^{1/\nu} + L^{-\omega}$$



3D classical clock model

- When $tL^{1/\nu}$ grows and $tL^{1/\nu_q'} \ll 1$, still dominated by the XY fixed point

Distance to x-axis: $d_X \propto L^{y_q}\left(1 + tL^{1/\nu}\right)$



$$d^X_{min} \propto t^{y_6 \nu} \propto t^{1.75(3)}$$

$$L_{d^X_{min}} \propto t^{-\nu} \propto t^{-0.6717(1)}$$

$$d^X_{min} \sim t^{1.88(2)}$$

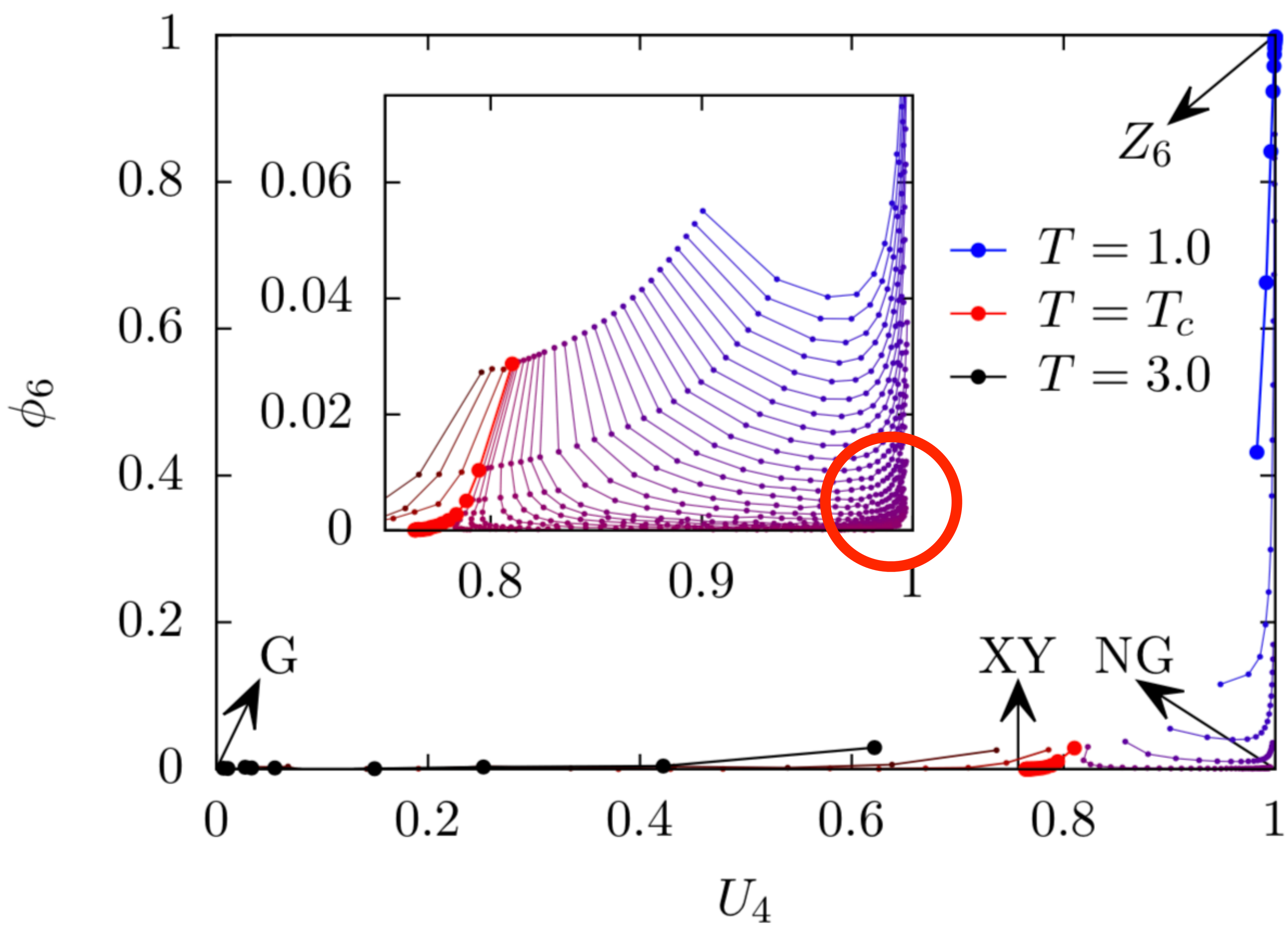
$$L_{d^X_{min}} \sim t^{-0.60(3)}$$

Shao , Guo and Sandvik, PRL (2020).

3D classical clock model

- When $tL^{1/\nu}$ is large enough and $tL^{1/\nu'_q} \ll 1$, namely $\xi \ll L \ll \xi'$

$$\phi_q \sim L^{-|y_q|}(tL^{1/\nu})^a \qquad 1 - U \propto (tL^{1/\nu})^{-r}$$



3D classical clock model

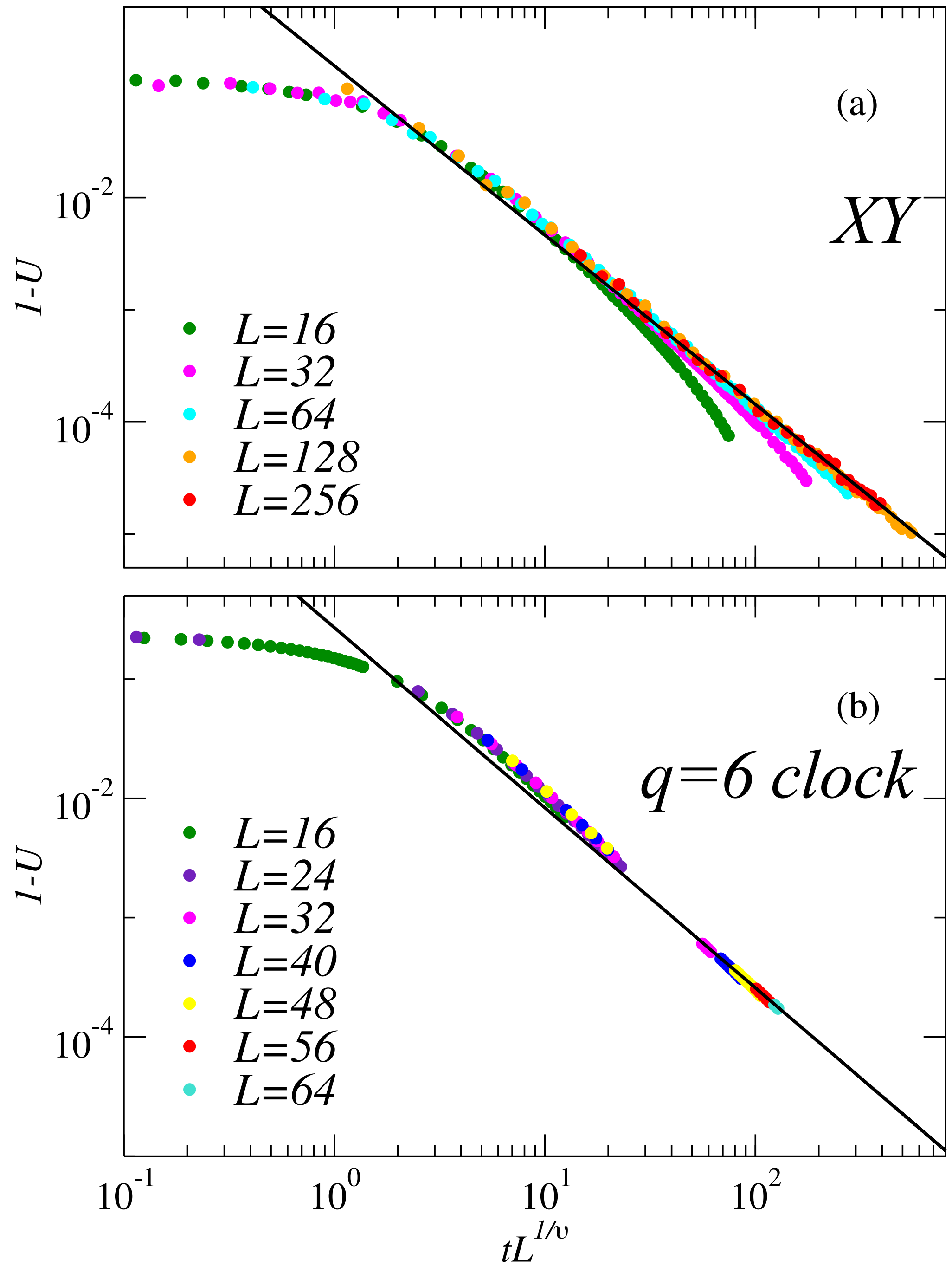
- When $tL^{1/\nu}$ is large enough and $tL^{1/\nu'_q} \ll 1$, namely $\xi \ll L \ll \xi'$

  - Asymptotic form:

$$1 - U \propto (tL^{1/\nu})^{-r}$$

No rigorous study, fit with MC data finds:

$$r = 1.52(2)$$



Shao , Guo and Sandvik, PRL (2020).

3D classical clock model

- When $tL^{1/\nu}$ is large enough and $tL^{1/\nu'_q} \ll 1$, namely $\xi \ll L \ll \xi'$

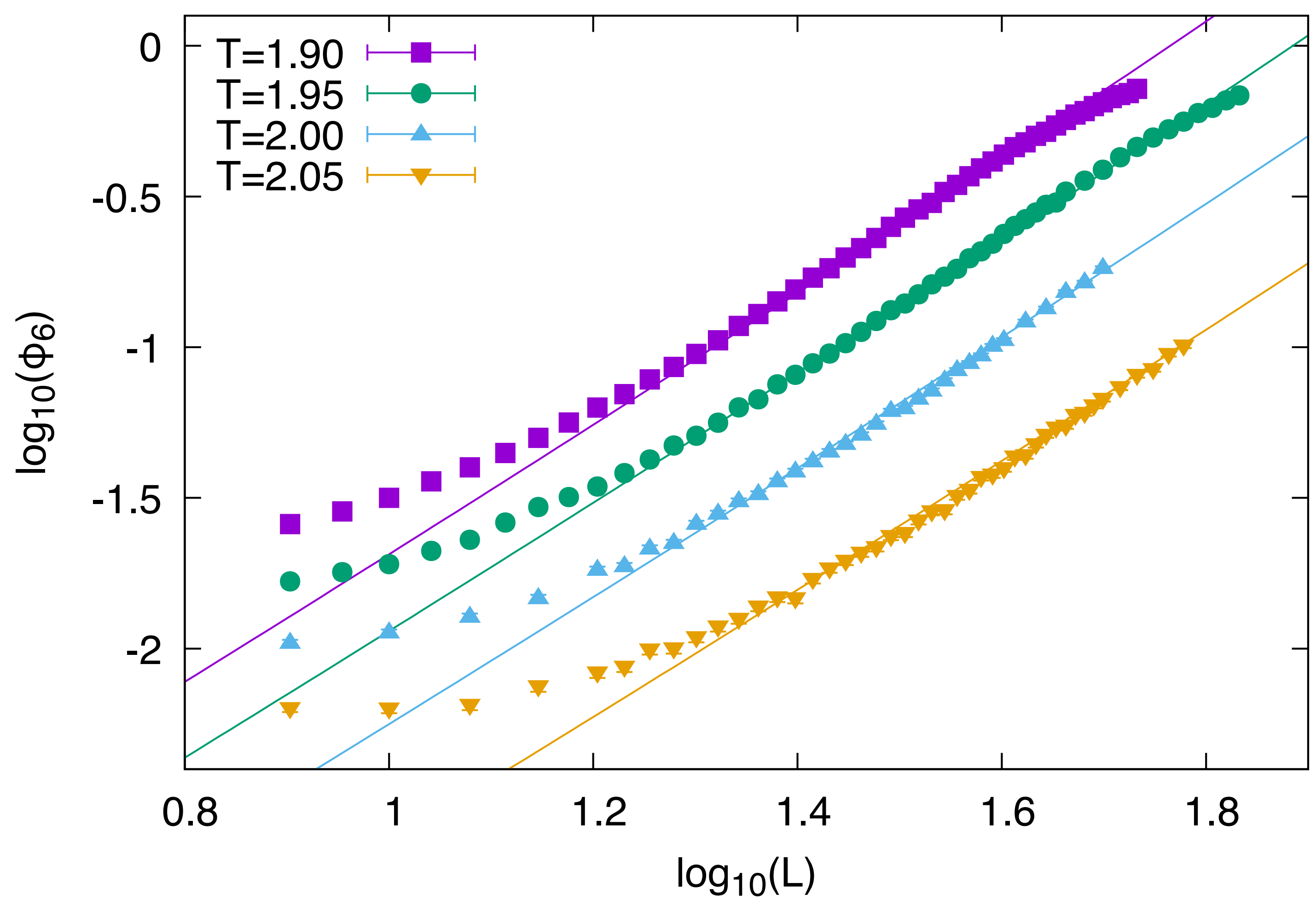
  - Scaling behavior of ϕ_q

  Proposals of the size dependent exist: $\phi_q \propto L^p$, based on different arguments, values of p are different (2 or 3).

$$\phi_q \sim L^{-|y_q|}(tL^{1/\nu})^a$$
$$\sim L^p t^{\nu(p+|y_q|)}$$

Fit with MC data show:

1) good agreement of the scaling form;
2) p=2.

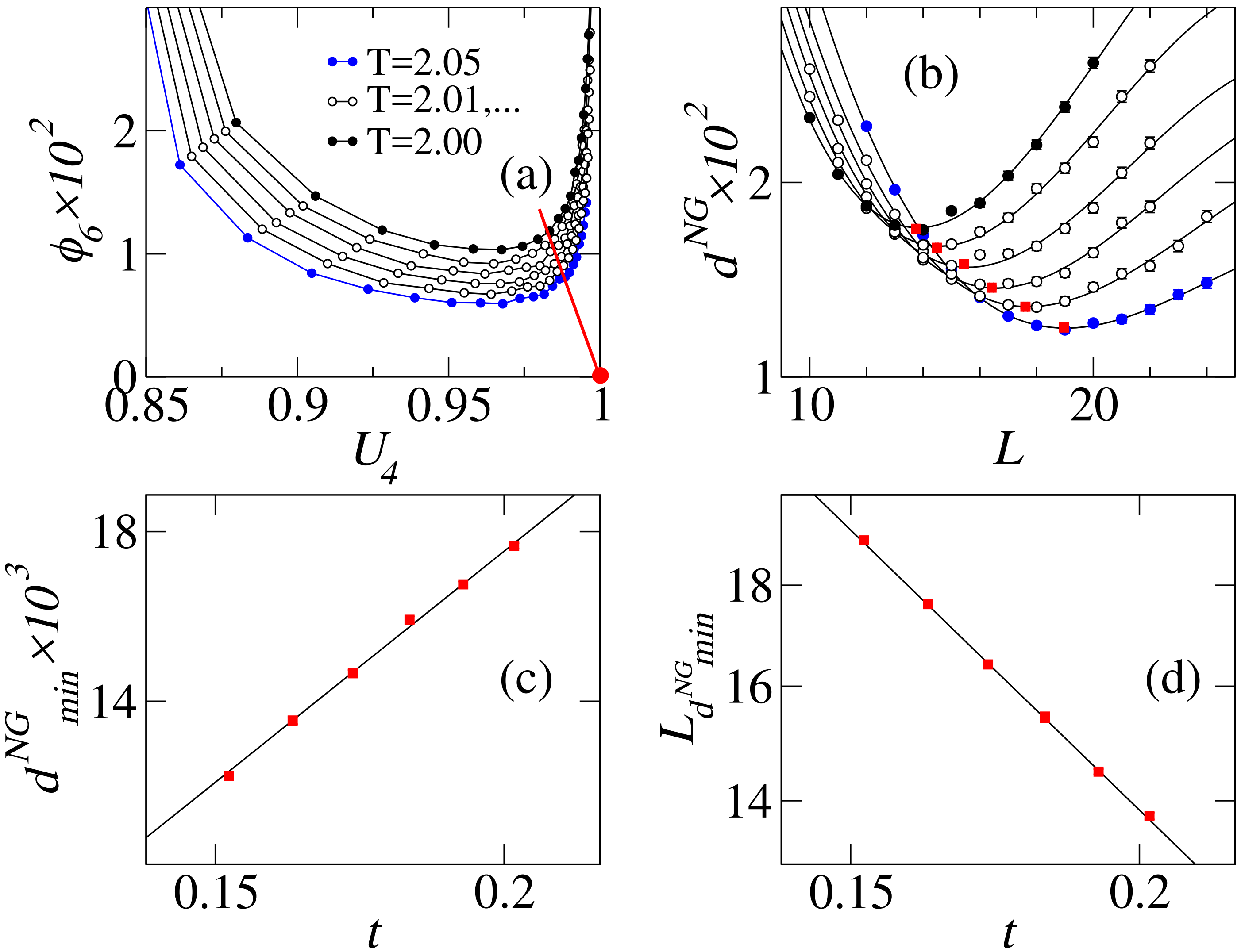


Patil , Shao, Sandvik , arXiv:2009.03249 (2020).

3D classical clock model

- When $tL^{1/\nu}$ is large enough and $tL^{1/\nu'_q} \ll 1$, namely $\xi \ll L \ll \xi'$
  - Distance to the NG fixed point

$$d_{NG} = \sqrt{L^{-2r/\nu}t^{-2r} + L^{2p}t^{2\nu(p-y_q)}}$$



$$d^{NG}_{\min} \propto f(t) \propto t^{0.9(1)}$$

$$L_{d^{NG}_{min}} \propto t^{-\nu(1-\frac{y_q}{r+2\nu})} \propto t^{-1.07(3)}$$

$$d^{NG}_{min} \sim t^{1.19(3)}$$

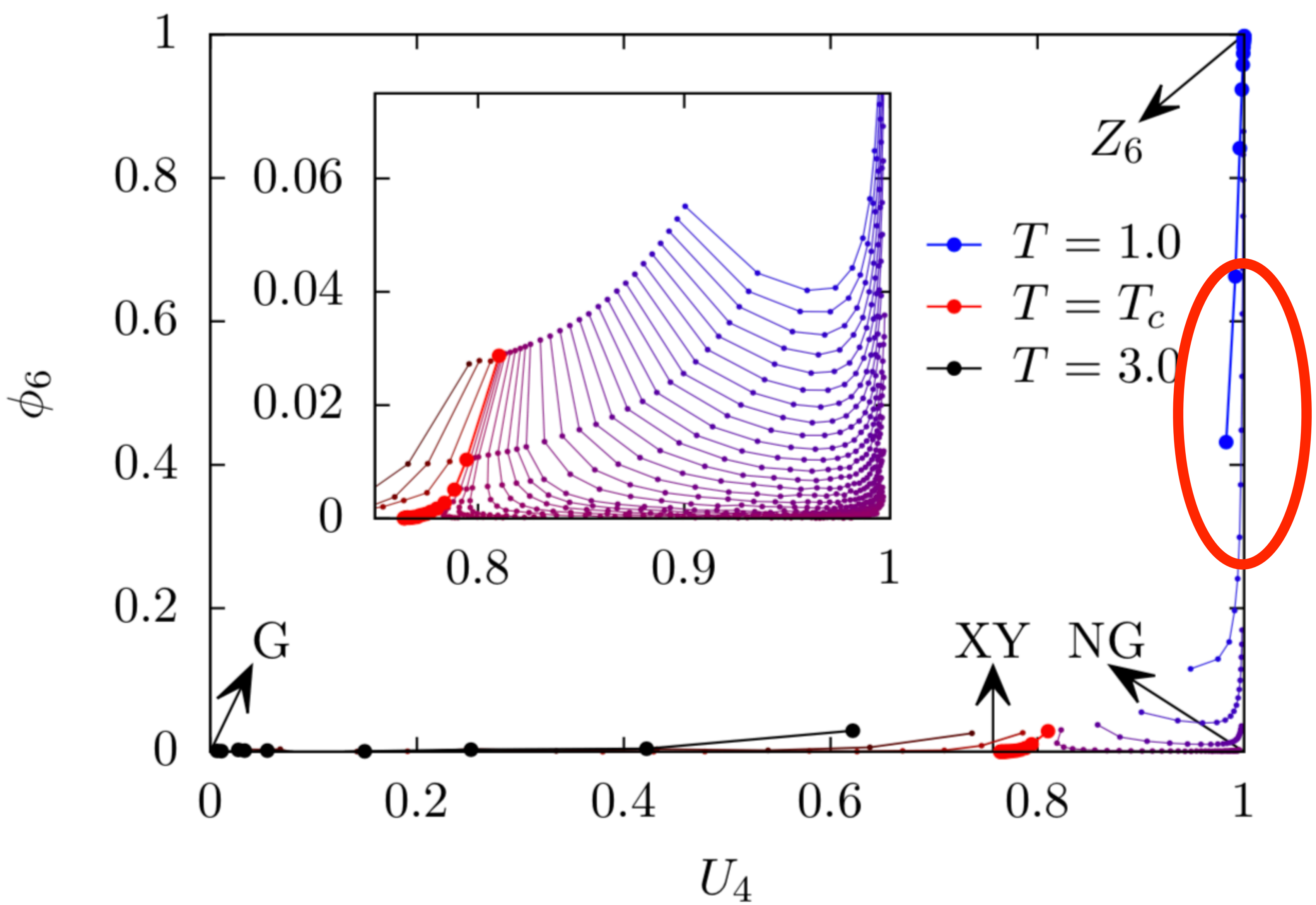
$$L_{d^{NG}_{min}} \sim t^{-1.14(2)}$$

Shao , Guo and Sandvik, PRL (2020).

3D classical clock model

- When $tL^{1/\nu_q'}$ starts to grow,

$$\phi_q \sim L^p t^{\nu(p+|y_q|)} g\left(tL^{1/\nu_q'}\right) \sim L^p t^{\nu(p+|y_q|)} \left(1 + tL^{1/\nu_q'}\right) \sim \phi_q(0) + tL^{1/\nu_q'}$$

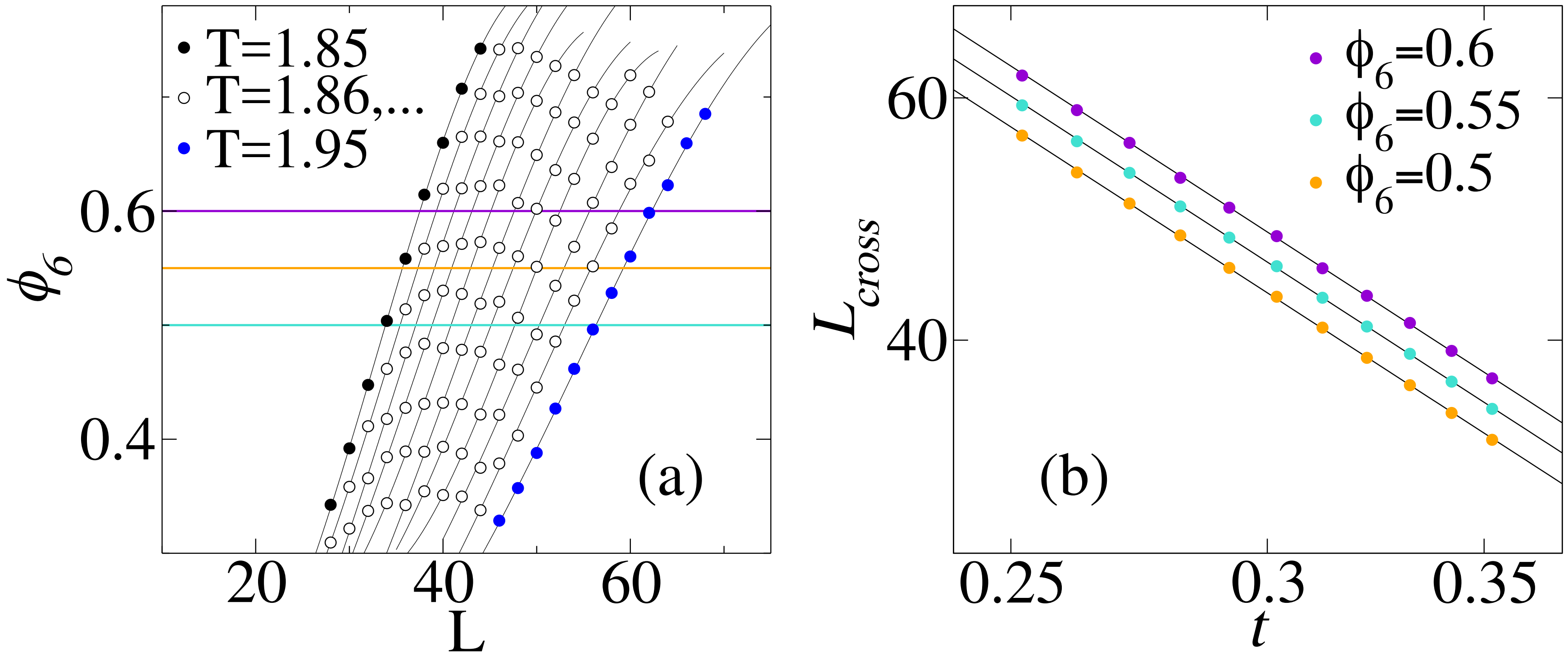


3D classical clock model

- When $tL^{1/\nu_q'}$ starts to grow

  For a given $\phi_q(cross)$ not close to 1, $\phi_q(cross) - \phi_q(0) \propto tL^{1/\nu'} \propto const.$

$$L_{cross} \propto t^{-\nu_q'}$$



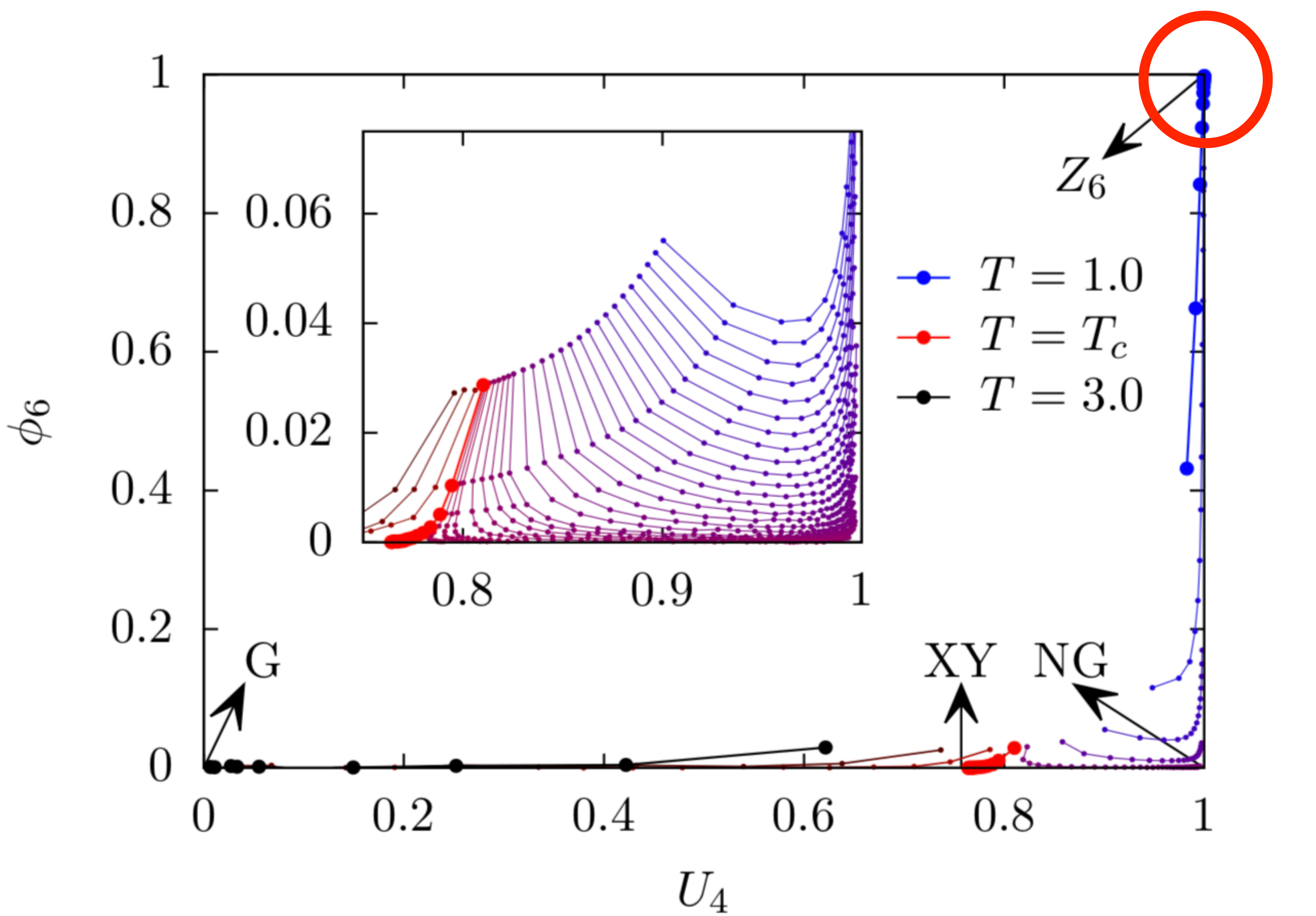
$$L_{cross} \sim t^{-1.52(1)}$$

Shao , Guo and Sandvik, PRL (2020).

3D classical clock model

- When tL^{1/ν'_q} >>1, i.e., $\phi_q \to 1$

$$\phi_q \sim L^p t^{\nu(p+|y_q|)} g\left(tL^{1/\nu'_q}\right) \sim L^p t^{\nu(p+|y_q|)} \left(tL^{1/\nu'_q}\right)^b$$



3D classical clock model

- When $tL^{1/\nu_q'}$ >>1, i.e., $\phi_q \to 1$

$$\phi_q \sim L^p t^{\nu(p+|y_q|)} g\left(tL^{1/\nu_q'}\right) \sim L^p t^{\nu(p+|y_q|)} (tL^{1/\nu_q'})^b$$

- Powers of L and t both *0*, leading to a scaling law:

$$\nu' = \nu\left(1 + \frac{|y_q|}{p}\right)$$

Also proposed in several theoretical studies.

- Values of the exponents agree with the scaling law.

$$\nu = 0.67175(1),\ \nu' = 1.52(4),\ y_q = 2.55(6),\ p = 2$$

Shao , Guo and Sandvik, PRL (2020).
Chubukov, et. al., PRB (1994). Okubo et. al., PRB (2015). Lonard et. At., PRL (2015).

2D quantum clock model

- Hamiltonian:

$$H = -s \sum_{\langle i,j \rangle} \cos(\theta_i - \theta_j) - (1-s) \sum_{i=1}^{N} Q_i$$

Choices of Q:

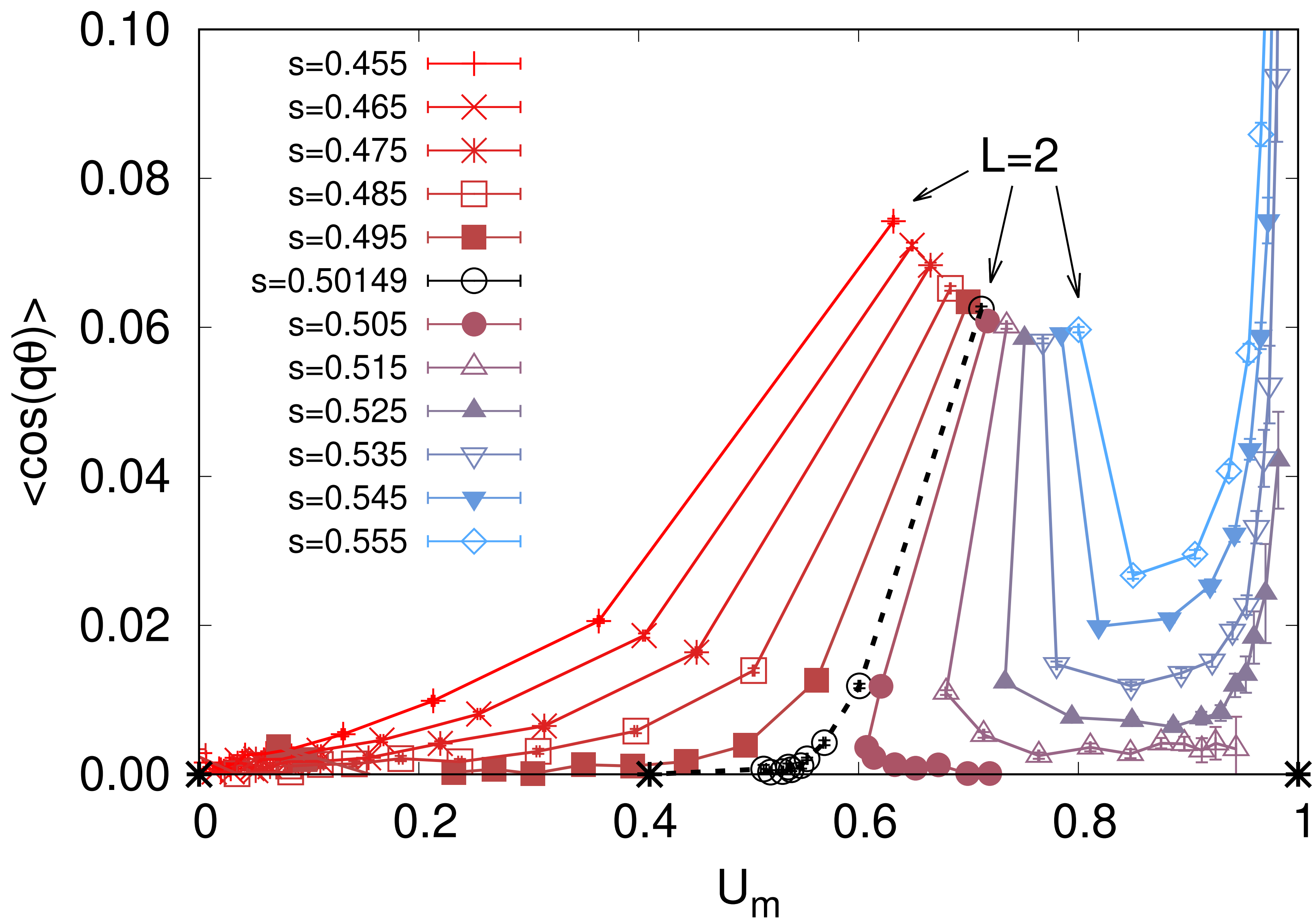
Model	⟨θi|Qi|θi′⟩	Constraint	Property
1	cos(θi – θi′) + 1	θi – θi′ = 2π/q	most clock-like
2	cos(θi – θi′) + 1	No	also clock-like
3	1/q	No	Potts-like

- All models found to have the 3D XY phase transition by SSE with cluster algorithm (See more details in app. B&C).

Patil , Shao, Sandvik , arXiv:2009.03249 (2020).

2D quantum clock model

- MC RG flow (Model 2, q=6)



Patil , Shao, Sandvik , arXiv:2009.03249 (2020).

2D quantum clock model

- Scaling dimensions of the Z(q) fields at S_c

Scaling with corrections:

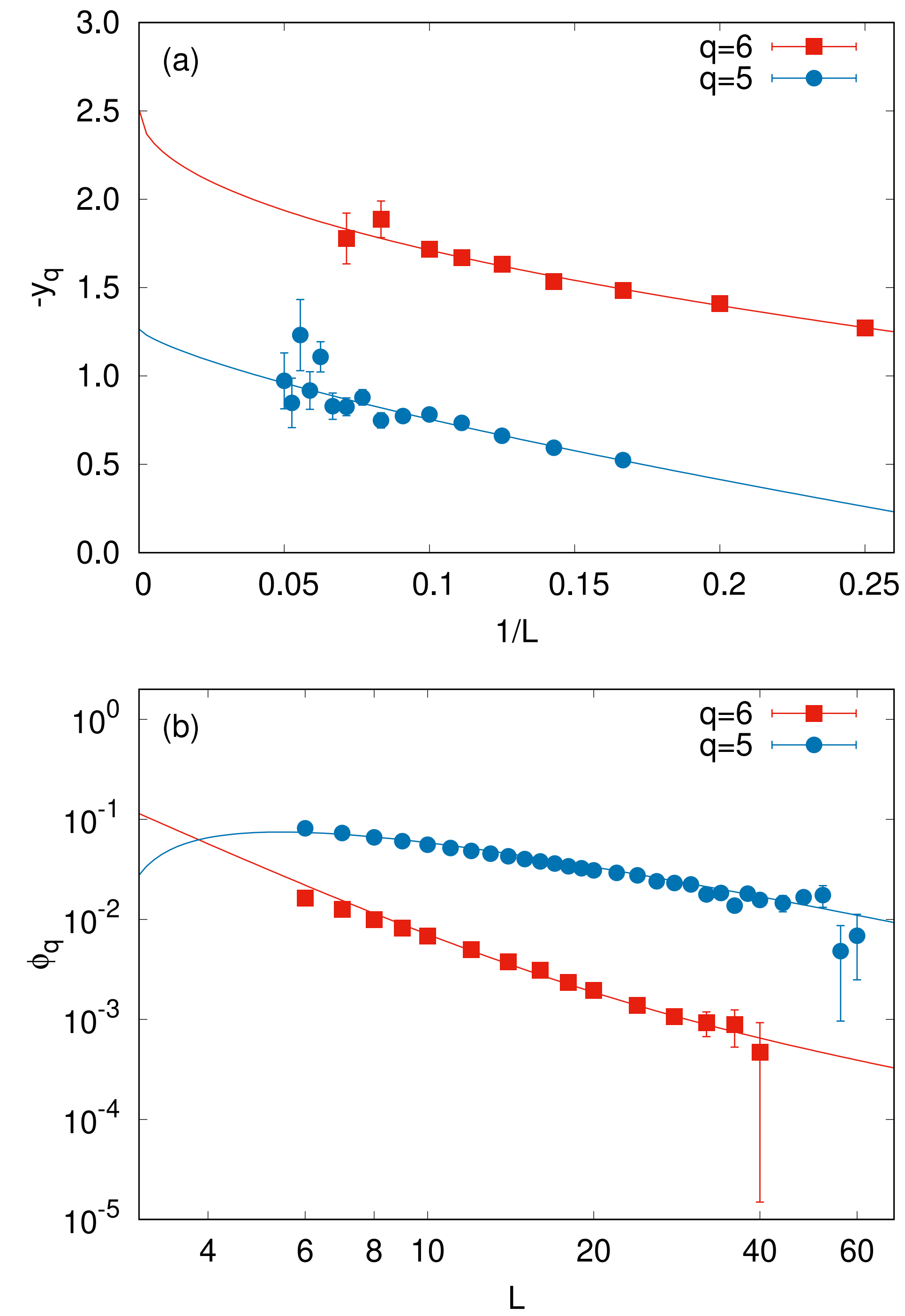
$$\phi_q = aL^{-|y_q|} + bL^{-c}$$

Flowing (L,2L) extrapolation:

$$-y_q(L) = \frac{1}{\ln(2)} \ln\left(\frac{\phi_q(L)}{\phi_q(2L)}\right)$$

$$-y_q(L) = |y_q| + eL^{-k}$$

Agrees with the known values.



Patil , Shao, Sandvik , arXiv:2009.03249 (2020).

2D quantum clock model

- When $\xi \ll L \ll \xi'$, relevant scaling behavior $\phi_q \sim L^p t^{\nu(p+|y_q|)}$
  - Different ϕ_q defined on:
    (averaged over rest direction(s))

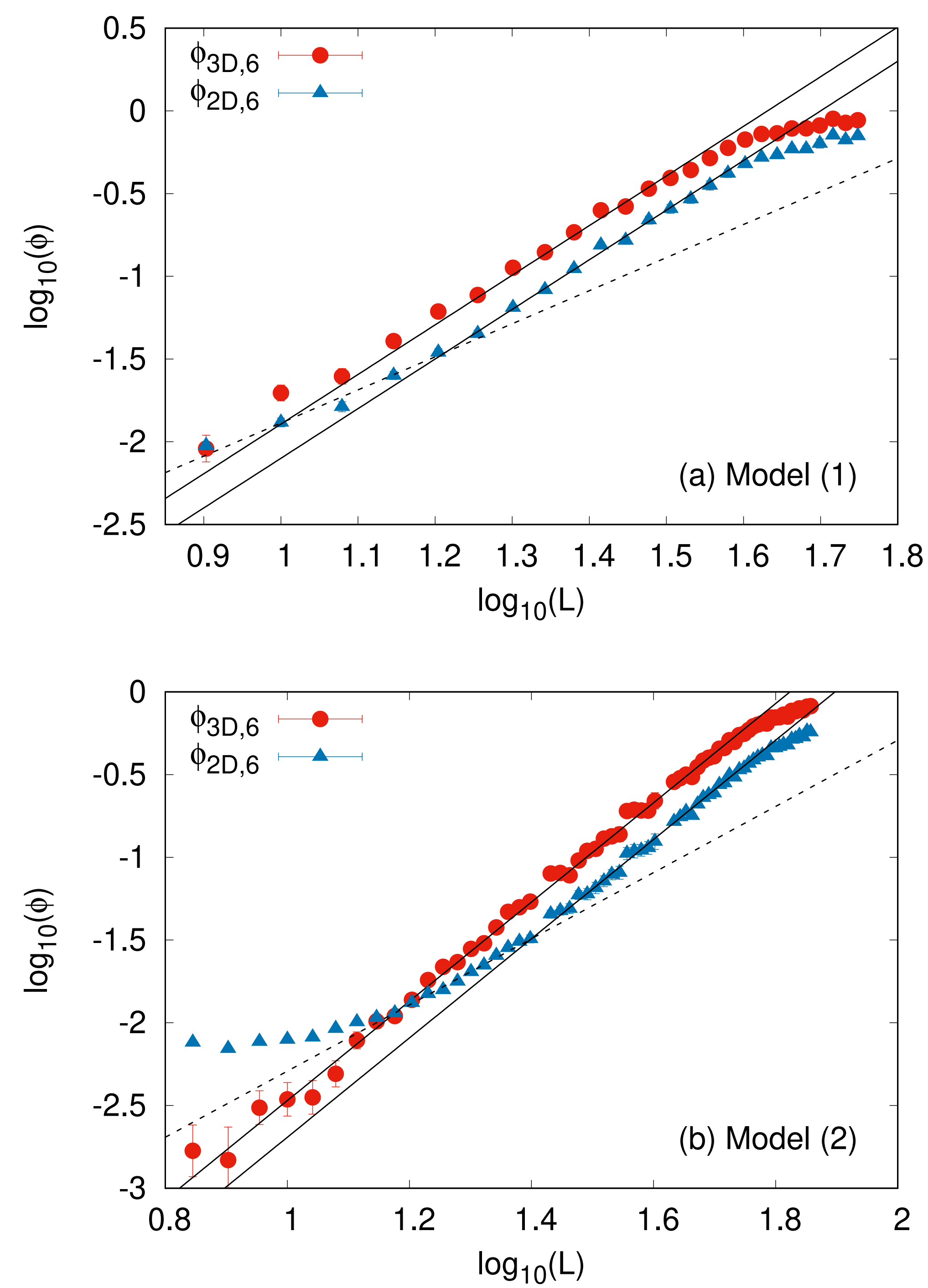
$$\vec{M}_{2D} = \frac{1}{L^2} \sum_{x,y} \vec{m}(x,y,\tau)$$

$$\vec{M}_{3D} = \frac{1}{\beta L^2} \int_0^\beta d\tau \sum_{x,y} \vec{m}(x,y,\tau)$$

$$\vec{M}_{1Ds} = \frac{1}{L} \sum_{x} \vec{m}(x,y,\tau)$$

$$\vec{M}_{1Dt} = \frac{1}{\beta} \int_0^\beta d\tau \vec{m}(x,y,\tau)$$

  - **p=3** found for all cases studied.

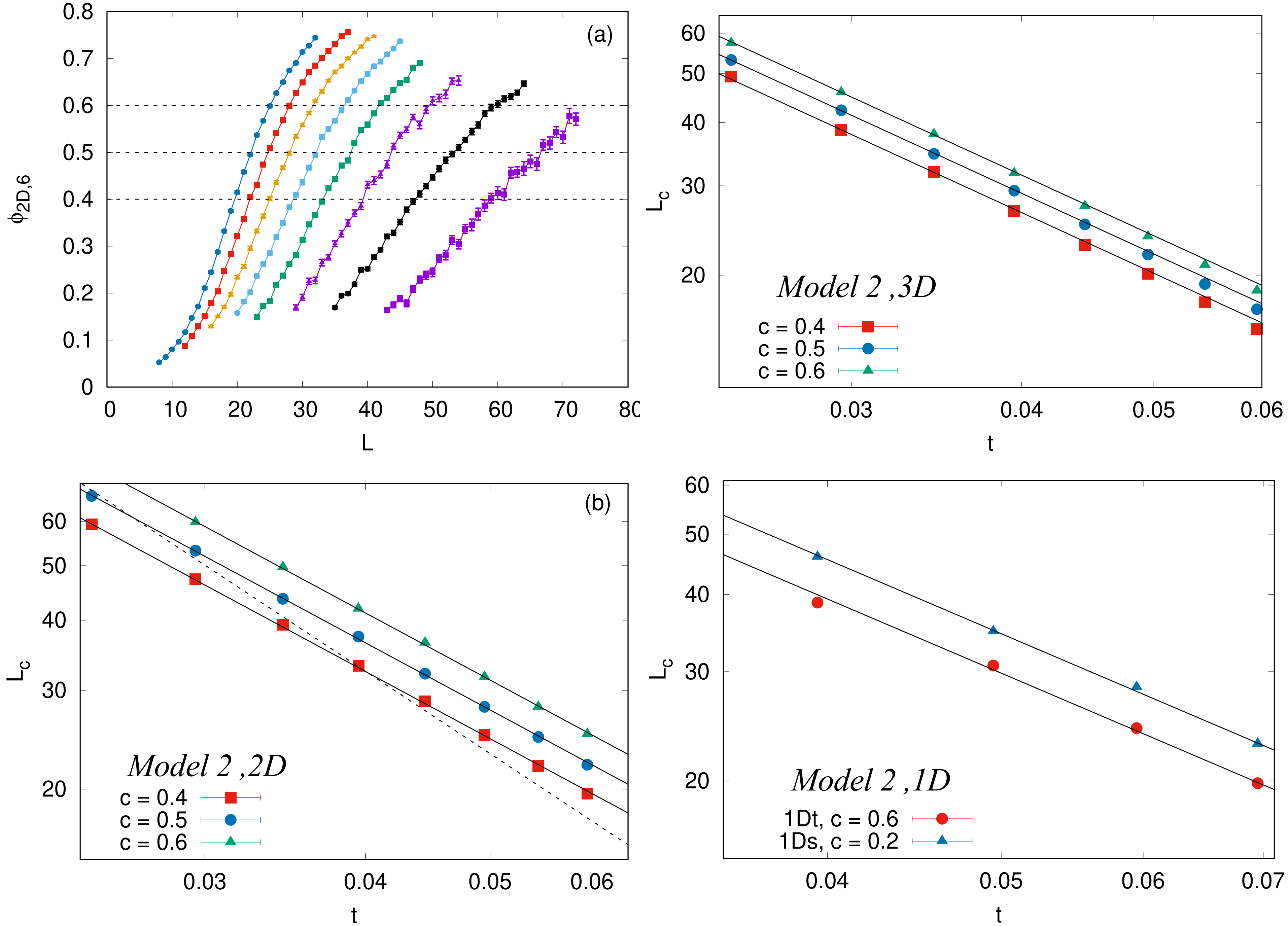


Patil , Shao, Sandvik , arXiv:2009.03249 (2020).

2D quantum clock model

- Extrapolation of ν'_q consistent with p=3 in the scaling law.

$$\nu' = \nu \left(1 + \frac{|y_q|}{p}\right)$$



Patil , Shao, Sandvik , arXiv:2009.03249 (2020).

3D anisotropic classical clock model

- D to D+1 mapping leads to the 3D anisotropic classical clock model (with strong anisotropy):

$$H_1 = -J_{\|} \sum_{\langle i,j \rangle_{\|}} \cos\left(\theta_i - \theta_j\right) - J_{\perp} \sum_{\langle i,j \rangle_{\perp}} \cos\left(\theta_i - \theta_j\right)$$

$$H_2 = -J_{\|} \sum_{\langle i,j \rangle_{\|}} \cos\left(\theta_i - \theta_j\right) - J_{\perp} \sum_{\langle i,j \rangle_{\perp}} \left(\delta_{\theta_i,\theta_j} - 1\right)$$

with $J_{\perp} = 1 + \lambda, \quad J_{\|} = 1 - \lambda$.

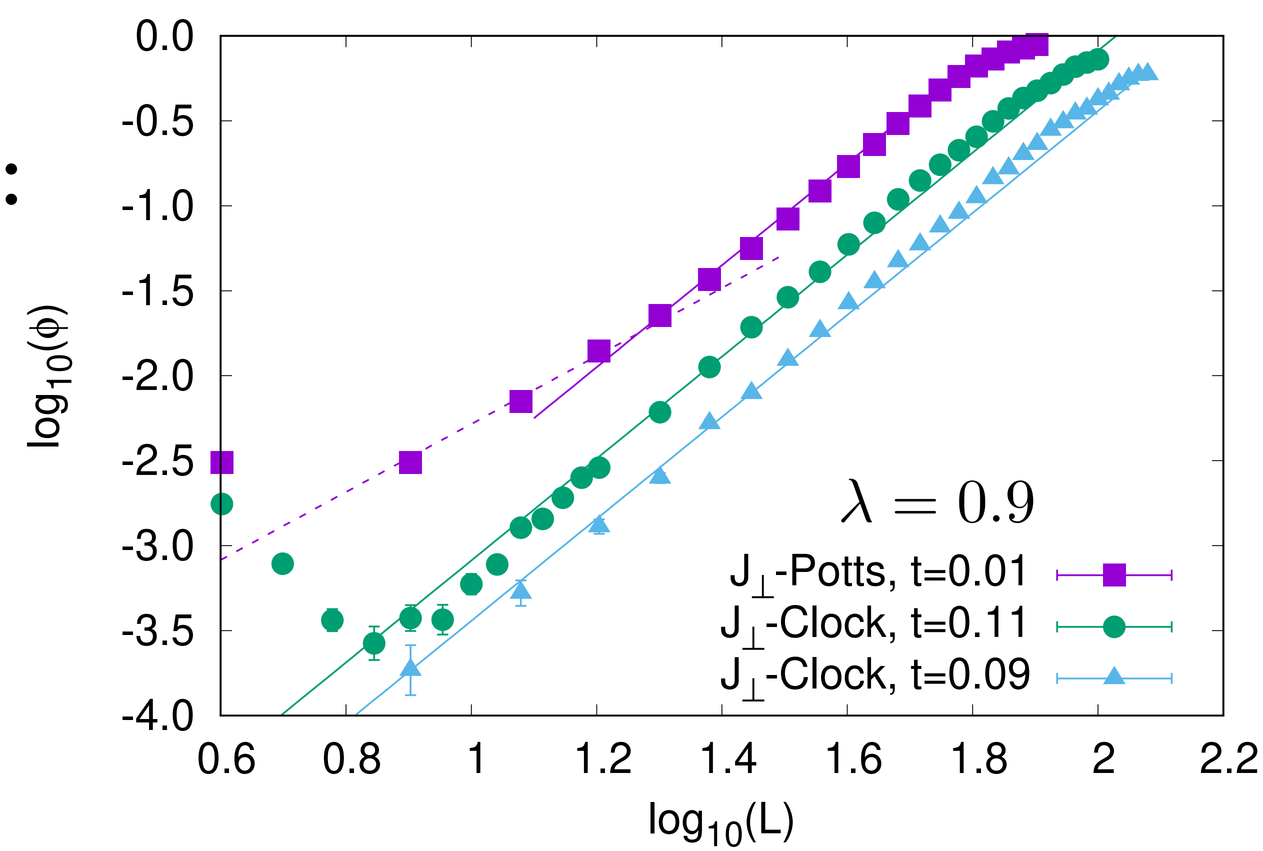
- Scaling near the NG fixed point:

When λ is large,

p=3 for $J_{\perp}$-Clock model;

crossover from p=2 to p=3

for $J_{\perp}$-Potts model.



Patil , Shao, Sandvik , arXiv:2009.03249 (2020).

3D anisotropic classical clock model

- Scaling near the NG fixed point:

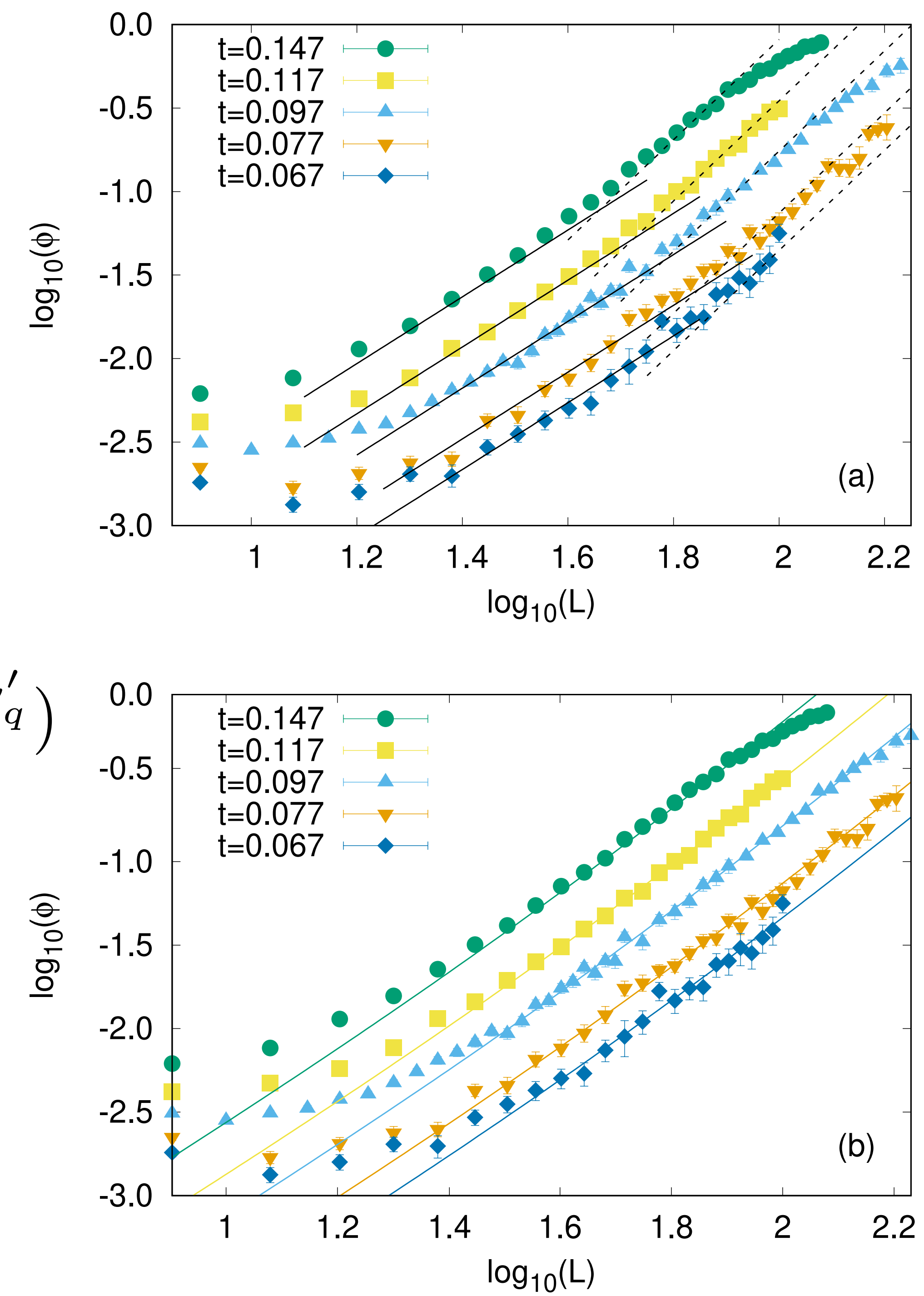
  -Crossover from p=2 to p=3 also found for $J_\perp$-Clock model.

  - Scaling with two different power laws:

$$\phi_q \sim (c_2 L^2 t^{\nu(2+|y_q|)} + c_3 L^3 t^{\nu(3+|y_q|)}) g(tL^{1/\nu'_q})$$

  has good agreement with the data.

$J_\perp$-Clock $\lambda = 0.5$



Patil , Shao, Sandvik , arXiv:2009.03249 (2020).

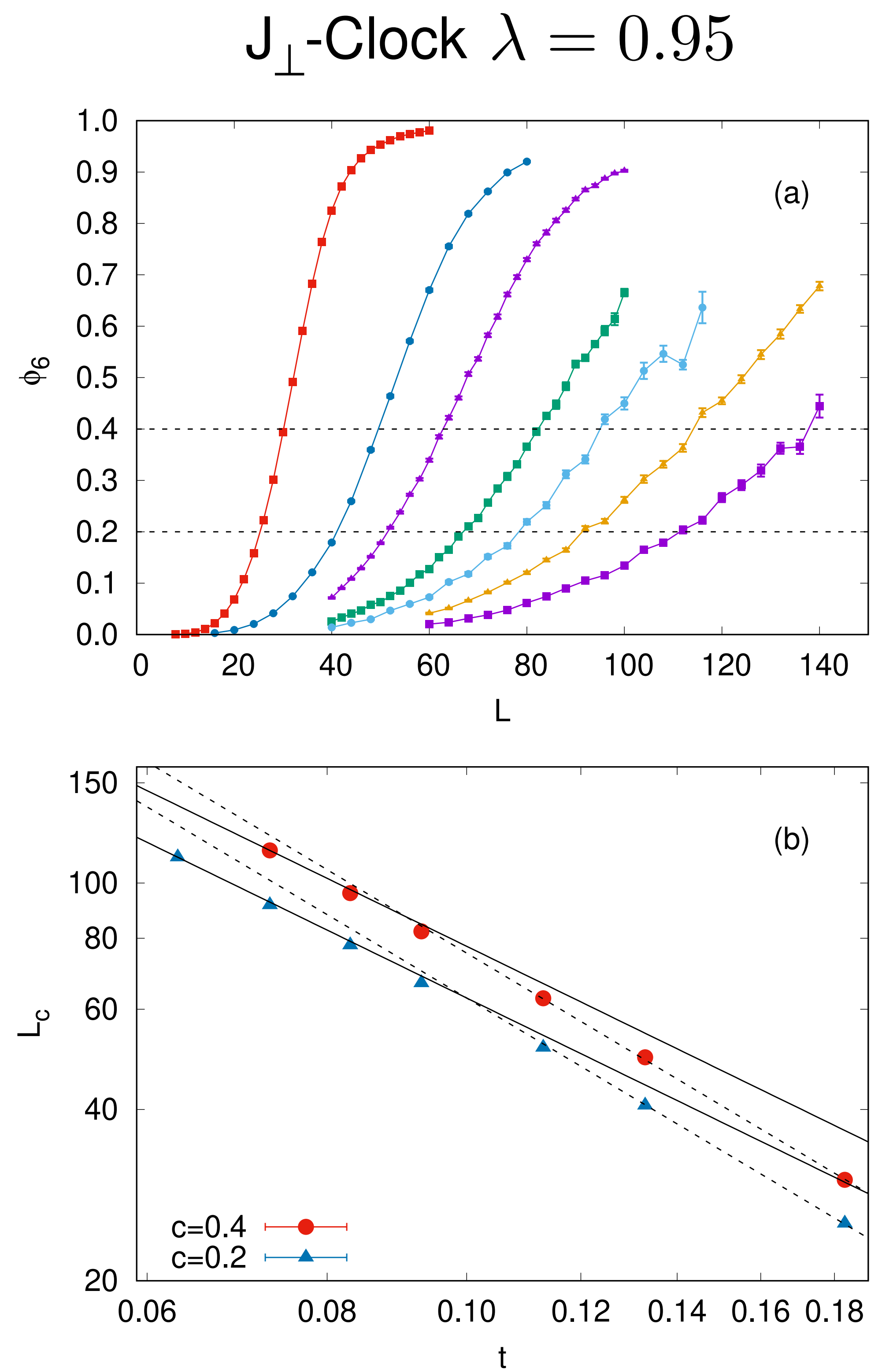
3D anisotropic classical clock model

- Evaluation of ν'_q

  - crossover found for large λ ;

  - $\nu'_q(p=2)$ behavior for larger t;

  - dominant behavior should always be $\nu'_q(p=3)$.



Patil , Shao, Sandvik , arXiv:2009.03249 (2020).

Conclusions

- Z(q) field in the clock model is a dangerously irrelevant field;
- Two length scale scaling function can describe all the renormalization stages;
- Unconventional U(1) to Zq cross-over found in quantum and anisotropic classical clock models, and further understanding needed for scaling power p=3;
- Discussions based on O(2) case should apply to O(n);
- Closely related to the deconfined quantum phase transition of the JQ model. (Anders on Friday)